521317S, Wireless Communications II

Final exam 16 May 2014. Leave a margin of two columns at the right-hand side of each page. Mark clearly where a solution to a problem ends and if it continues on a following page or paper. Use of pencil in the solutions is allowed.

1. Consider a time-invariant single-user channel

$$\mathbf{y}[m] = \mathbf{h}x[m] + \mathbf{w}[m] \tag{1}$$

where $\mathbf{h} = [h_1, \ldots, h_L] \in \mathbb{C}^L$ is the fixed channel vector between the transmitter and L receive antennas, $x_k[m]$ is the transmitted symbol at time instant m, subject to $\mathbb{E}[|x|^2] \leq P$, and $\mathbf{w}[m]$ is the received noise vector where each $w_l[m] \sim \mathcal{CN}(0, N_0)$.

1.1 Draw a figure illustrating the system model in (1).

1.2 Assuming $L = 1$ and $h_1 = 1$, express the capacity (bits/s) across the bandwidth W. Explain with proper illustrations the meaning of 'power limited region' and 'bandwidth limited region'.

1.3 Explain the meaning of receive beamforming across L antennas. Derive the signal-to-noise ratio at the receiver output and proof (by Cauchy-Schwarz inequality) that the maximum ratio combining (MRC) is the capacity optimal reception strategy. Finally, give the capacity expression (bits/s/Hz) for the system model in (1).

1.4 Consider the system model in (1) with

$$h_l = a \exp\left(-\frac{j2\pi d_l}{\lambda_c}\right)$$

where a is the common amplitude for all antennas, d_l is the distance between the transmit antenna and the lth receive antenna, and λ_c is the carrier wavelength. Let us denote Δ_r as the normalised receive antenna separation and assume $d_l >> \Delta_r \lambda_c$. How is the optimal MRC receiver simplified in this case? What is the corresponding capacity expression.

2. Assume time-invariant uplink channel with a single BS with n_r receive antennas and K users, where each user k is equipped with n_{t_k} transmit antennas. The received signal vector at symbol time m is

$$\mathbf{y}[m] = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k[m] + \mathbf{n}[m] \tag{2}$$

where $\mathbf{x}_k[m]$ is the TX vector of user k at time instant m, subject to $\mathbb{E}[\mathrm{Tr}(\mathbf{x}_k \mathbf{x}_k^\mathrm{H})] = \mathrm{Tr}(\mathbf{K}_{x_k}) \leq P_k$, $\mathbf{y} \in \mathbb{C}^{n_\mathrm{r}}$ is the RX signal, $\mathbf{w} \sim \mathcal{CN}(0, N_0\mathbf{I})$ is complex white Gaussian noise, and $\mathbf{H}_k \in \mathbb{C}^{n_\mathrm{r} \times n_{\mathrm{t}_k}}$ is the channel matrix of user k.

2.1 Draw a figure illustrating the system model in (2).

2.2 Assuming $K = 1$, and CSI knowledge only at the receiver ($\mathbf{K}_{x_1} = \frac{P}{n_{t_1}}\mathbf{I}_{n_{t_1}}$), show that MMSE-SIC is the capacity achieving receiver architecture, i.e.,

$$\sum_{i=1}^{n_{\min}} \log(1 + \gamma_i^{\text{mmse-sic}}) = \log\det(\mathbf{I}_{n_r} + \frac{P}{n_{t_1}N_0}\mathbf{H}_1\mathbf{H}_1^{\text{H}}) \tag{3}$$

where $\gamma_i^{\text{mmse-sic}}$ is the SINR of the stream i at the output of the MMSE-SIC receiver, $n_{\min} = \min(n_{t_1}, n_r)$ and $\mathbf{H}_1 = [\mathbf{h}_{1,1}, \ldots, \mathbf{h}_{1,n_{t_1}}]$. *Hint: for* $\mathbf{a} \in \mathbb{C}^{m\times 1}$, $\mathbf{b} \in \mathbb{C}^{1\times m}$, *and invertible* $\mathbf{R} \in \mathbb{C}^{m\times m}$, $\log\det(\mathbf{R} + \mathbf{ab}) = \log\det(\mathbf{R}) + \log(1 + \mathbf{b}\mathbf{R}^{-1}\mathbf{a})$.

2.3 For $K = 3$, let $\{n_{t_1}, n_{t_2}, n_{t_3}\} = \{1, 2, 3\}$, $n_r = 6$ and $P_k = P, \forall\, k$. How many streams with non-zero power are allocated per user both at low and high SNR? Assume *full CSIT* knowledge at all nodes. Justify your answer.

2.4 For $K = 2$ and full CSIT, derive and depict the capacity region of the system in (2) assuming Gaussian input distribution. *Hint: The capacity region is a union of pentagons for all feasible* $\mathbf{K}_{x_k}, k = 1, 2$.

Hint: $h(\mathbf{y}) \leq \log\det(\pi e\mathbb{E}[\mathbf{y}\mathbf{y}^{\text{H}}])$, $I(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = I(\mathbf{x}_1; \mathbf{y}) + I(\mathbf{x}_2; \mathbf{y}|\mathbf{x}_1) + I(\mathbf{x}_3; \mathbf{y}|\mathbf{x}_1, \mathbf{x}_2)$

3. Consider time-invariant downlink channel with 3 single-antenna users and a single BS with n_t transmit antennas. The received signal vector $y_k \in \mathbb{C}$ for user k at symbol time m is described by

$$y_k[m] = \sum_{i=1}^{3} \mathbf{h}_k^{\text{H}}\mathbf{u}_i x_i[m] + w_k[m] \tag{4}$$

where $x_k = \sqrt{p_k}d_k$ is the TX symbol of user k split into the normalised data symbol $d_k \in \mathbb{C}$ ($\text{E}[|d_k|^2] = 1$) and the corresponding power allocation p_k, $\mathbf{u}_k \in \mathbb{C}^{n_t}$ is the normalised beamformer, $\|\mathbf{u}_k\| = 1$, $w_k \sim \mathcal{CN}(0, N_0)$ is the complex white Gaussian noise and $\mathbf{h}_k \in \mathbb{C}^{n_t}$ is the channel vector of user k ideally known at the transmitter.

Assume the BS applies non-linear Costa (dirty paper) precoding with user encoding order $1, 3, 2$. Find the sum rate optimal precoders $\mathbf{m}_k = \sqrt{p_k}\mathbf{u}_k, k = 1, 2, 3$.

Hint: Start with dual uplink formulation with reverse decoding order, find the sum rate optimal powers and the MMSE-SIC receivers in the dual uplink, and apply the uplink-downlink duality to find the corresponding downlink precoders.