Statistical Signal Processing 1, Final exam, 8-Jan-2024, 16:20-19:00

INSTRUCTIONS

Solve all five (5) problems. No material is allowed (no calculator etc.) except for two handwritten A4 papers (totally 4 A4 pages). Please make sure to return your solutions on time. Write your name and your student ID on all papers. Clearly indicate which solution you are solving. Intermediate steps need to be included (do not just give the answer). Simplify your solutions to the extent possible. Return the exam answer paper(s) to the invigilator before leaving the exam hall. Show the student card (or mobile student card on the mobile phone) or identity card (passport, ID card, or KELA card with a photo) to the exam invigilator.

A. *Problem 1*

Assume that received samples $x[n] = \theta_1 + \theta_2 n + \theta_3 \cos(n) + w[n]$, $n = 1, 2, 3, 4$. Here $\theta_1, \theta_2, \theta_3$ are the unknown parameters to be estimated, $w[n]$ is white Gaussian noise and n is the time index. Let us define $\boldsymbol{\theta}$ as a vector containing the unknown parameters

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

and let us collect the received samples to a vector

$$\mathbf{x} = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

Find observation matrix $\mathbf{H}$ and vector $\mathbf{w}$ such that $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$.

Solution: The observation matrix is:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cos(1) \\ 1 & 2 & \cos(2) \\ 1 & 3 & \cos(3) \\ 1 & 4 & \cos(4) \end{bmatrix}$$

and the the vector $\mathbf{w}$ is:

$$\mathbf{w} = \begin{bmatrix} w[1] \\ w[2] \\ w[3] \\ w[4] \end{bmatrix}$$

B. *Problem 2*

The data

$$x[n] = 2A + w[n]$$

for $n = 0, 1, \cdots, N-1$ is observed where $w[n]$ is zero-mean white Gaussian noise with variance σ^2. We want to estimate unknown parameter A based on the observations $x[n]$.

1) Find the CRLB for A
2) Find the efficient estimator and verify that it is unbiased and reaches the CRLB.

Solution: By using the theory of signal in white Gaussian noise:

$$\mathrm{CRLB} = \frac{\sigma^2}{\sum\limits_{n=0}^{N-1}\left(\frac{\partial(2A)}{\partial A}\right)^2} = \frac{\sigma^2}{\sum\limits_{n=0}^{N-1}(2)^2} = \frac{\sigma^2}{4N}$$

By using the standard approach:

$$\begin{aligned}
&p(\mathbf{X}) = \tfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\tfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A)^2\right)\\
&\log p(\mathbf{X}) = -\tfrac{N}{2}\log\left(2\pi\sigma^2\right) - \tfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A)^2\\
&\tfrac{\partial \log p(\mathbf{X})}{\partial A} = \tfrac{2}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A)\\
&\tfrac{\partial^2 \log p(\mathbf{X})}{\partial A^2} = -\tfrac{4N}{\sigma^2}\\
&\mathrm{CRLB} = \tfrac{\sigma^2}{4N}
\end{aligned}$$

To get efficient estimator, we use the first derivative:

$$\begin{aligned}
&\tfrac{\partial \log p(\mathbf{X})}{\partial A} = \tfrac{2}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A) = \tfrac{2}{\sigma^2}\sum_{n=0}^{N-1}x[n] + \tfrac{2}{\sigma^2}\sum_{n=0}^{N-1}-2A\\
&= \tfrac{2}{\sigma^2}\sum_{n=0}^{N-1}x[n] - \tfrac{4AN}{\sigma^2} = \tfrac{4N}{\sigma^2}\left(\tfrac{1}{2N}\sum_{n=0}^{N-1}x[n] - A\right)
\end{aligned}$$

Thefore, we get that the efficient estimator is

$$\widehat{A} = \frac{1}{2N}\sum_{n=0}^{N-1}x[n]$$

Let us check it is unbiased:

$$E\left[\widehat{A}\right] = \frac{1}{2N}\sum_{n=0}^{N-1}E\left[x[n]\right] = \frac{1}{2N}\sum_{n=0}^{N-1}2A = \frac{2AN}{2N} = A$$

Let us check it reaches the CRLB:

$$\mathrm{Var}\left[\hat{A}\right] = \frac{1}{4N^2}\sum_{n=0}^{N-1}\mathrm{Var}\left[x[n]\right] = \frac{1}{4N^2}\sum_{n=0}^{N-1}\sigma^2 = \frac{\sigma^2 N}{4N^2} = \frac{\sigma^2}{4N}$$

C. Problem 3

There are conditionally independent measurements $z[1], z[2], \cdots, z[N]$ which follow the distribution

$$p\left(z[k] \mid \lambda\right) = \frac{\lambda^{z[k]}\exp(-\lambda)}{z[k]!}$$

where $z[k] \geq 0$. Furthermore, it is known that

$$p(\lambda) = \exp(-\lambda)$$

where $\lambda > 0$. Based on the N measurements, find the MAP estimate of λ.

Solution: The conditional PDF of the vector of observations $\mathbf{Z}$ is

$$p\left(\mathbf{Z}|\,\lambda\right) = \frac{\lambda^{\sum\limits_{k=1}^{N} z[k]} \exp\left(-N\lambda\right)}{\prod\limits_{k=1}^{N} z[k]!}$$

The MAP estimate is found with

$$\hat{\lambda} = \arg\max_{\lambda} p\left(\mathbf{Z}|\,\lambda\right) p\left(\lambda\right) = \arg\max_{\lambda} \frac{\lambda^{\sum\limits_{k=1}^{N} z[k]} \exp\left(-\left(N+1\right)\lambda\right)}{\prod\limits_{k=1}^{N} z[k]!}$$

To solve this, let us take the logarithm

$$\log\left(p\left(\mathbf{Z}|\,\lambda\right) p\left(\lambda\right)\right) = \sum_{k=1}^{N} z[k] \log\left(\lambda\right) - \left(N+1\right)\lambda - \sum_{k=1}^{N} \log\left(z[k]!\right)$$

and next take the derivative

$$\frac{\partial \log\left(p\left(\mathbf{Z}|\,\lambda\right) p\left(\lambda\right)\right)}{\partial\lambda} = \frac{\sum\limits_{k=1}^{N} z[k]}{\lambda} - \left(N+1\right)$$

Setting the derivative to zero and solving we get the MAP estimate (since the second derivative is always negative):

$$\hat{\lambda} = \frac{\sum\limits_{k=1}^{N} z[k]}{N+1}$$

D. *Problem 4*

Assume the following PDF parameterized by a:

$$f\left(x;a\right) = \begin{cases} ax\exp\left(-\frac{ax^2}{2}\right) & x>0, a>0 \\ 0 & \text{otherwise} \end{cases}$$

This is a valid PDF which can be proven by integrating it by using the identity

$$\int ax\exp\left(-\frac{ax^2}{2}\right)dx = -\exp\left(-\frac{ax^2}{2}\right)$$

Assume two hypothesis H_0 and H_1 and one measurement x.

$$\begin{aligned} H_0 &: f(x) = f\left(x;a_0\right) \\ H_1 &: f(x) = f\left(x;a_1\right) \end{aligned}$$

where $a_1 < a_0$. So for example under hypothesis H_0 the parameter $a = a_0$.

- Based on the measured value x, find the Neyman-Pearson detector (including deriving its threshold for the target probability of false alarm).
- Find the probability of detection as the function of the probability of false alarm.
- Assume (only for this subquestion) target probability of false alarm is $\alpha = 0.01$, $a_0 = 10$, and $a_1 = 5$. Find the numerical value of the obtained probability of detection (can be solved without calculator).

The problem is chosen in such way that we avoid the case where one hypothesis would be always chosen regardless of the value of x.

Solution: The Neyman-Pearson test is

$$\frac{a_1 x \exp\left(-\frac{a_1 x^2}{2}\right)}{a_0 x \exp\left(-\frac{a_0 x^2}{2}\right)} > \gamma$$

which can be simplified

$$\exp\left(\frac{a_0 - a_1}{2} x^2\right) > \frac{\gamma a_0}{a_1}$$

and finally we get

$$x^2 > \frac{2 \log\left(\frac{\gamma a_0}{a_1}\right)}{a_0 - a_1} = \gamma'$$

which can be written as (assuming right hand side is positive, which we can assume since it was said that one hypothesis is not always chosen and also since x is assumed always positive)

$$x > \gamma''$$

Assume that target probability of false alarm is α. Now we need that

$$P\left(x > \gamma''; H_0\right) = \int_{\gamma''}^{\infty} a_0 x \exp\left(-\frac{a_0 x^2}{2}\right) dx = \exp\left(-\frac{a_0 (\gamma'')^2}{2}\right) = \alpha$$

By solving this we get as the threshold

$$\gamma'' = \sqrt{\frac{-2\log(\alpha)}{a_0}}$$

The probability of detection is obtained as

$$P_D = P\left(x > \sqrt{\frac{-2\log(\alpha)}{a_0}}; H_1\right) = \int_{\sqrt{\frac{-2\log(\alpha)}{a_0}}}^{\infty} a_1 x \exp\left(-\frac{a_1 x^2}{2}\right) dx = \exp\left(\frac{a_1}{a_0}\log(\alpha)\right) = \alpha^{\left(\frac{a_1}{a_0}\right)}$$

For the last question: we directly use the above result to get that probability of detection is 0.1.

E. Problem 5

Consider the observations following the model for observations $Y[k]$

$$Y[k] = \theta + w[k]$$

for $k = 0, 1, \cdots, N-1$. We want to estimate the unknown parameter θ. The zero-mean noise samples $w[k]$ are independent and have the PDF

$$p(x) = \frac{1}{2}\exp(-|x|)$$

for $-\infty < x < \infty$. Find all maximum likelihood estimates of θ when $N = 2$. Hint: Since noise is zero-mean, a very reasonable estimator (and valid ML estimate for $N = 2$) would be sample mean. You need to find all solutions in addition to the sample mean.

1) Solution: First we notice that the PDF of the observed samples is translation (shift) of the PDF of the noise

$$f\left(Y\left[k\right];\theta\right)=\frac{1}{2}\exp\left(-\left|Y\left[k\right]-\theta\right|\right)$$

Therefore, the joint PDF for $N=2$ is

$$f\left(\mathbf{Y};\theta\right)=\frac{1}{4}\exp\left(-\left|Y\left[0\right]-\theta\right|\right)\exp\left(-\left|Y\left[1\right]-\theta\right|\right)$$

Without loss of generality, let us sort the samples such that $Y[0]\leq Y[1]$. Now we have three possibilities for the location of θ. It can be less than $Y[0]$ (case 1), between $Y[0]$ and $Y[1]$ (case 2), or greater than $Y[1]$ (case 3). We now get the joint PDF for $N=2$ with

$$f\left(\mathbf{Y};\theta\right)=\begin{cases}\frac{1}{4}\exp\left(-Y[0]-Y[1]+2\theta\right) & \theta<Y\left[0\right]\\ \frac{1}{4}\exp\left(Y[0]-Y[1]\right) & Y\left[0\right]\leq\theta<Y\left[1\right]\\ \frac{1}{4}\exp\left(Y[0]+Y[1]-2\theta\right) & \theta\geq Y\left[1\right]\end{cases}$$

Let us do ML estimate for each case separately and then combine the results (choose to one that leads to maximum likelihood). For case 1, joint PDF is largest when θ is largest possible for that case

$$\hat{\theta}_1=Y\left[0\right]$$

The joint PDF at this θ is

$$\frac{1}{4}\exp\left(Y[0]-Y[1]\right)$$

For case 3, smallest θ gives the largest joint PDF so that

$$\hat{\theta}_3=Y\left[1\right]$$

For this case, the joint PDF at this θ

$$\frac{1}{4}\exp\left(Y[0]-Y[1]\right)$$

For case 2, any value between $Y[0]$ and $Y[1]$ is valid and the joint PDF is

$$\frac{1}{4}\exp\left(Y[0]-Y[1]\right)$$

We noticed that all above joint PDFs are equal. So any value between (including the end points) $Y[0]$ and $Y[1]$ can be chosen as the ML estimate. Obviously this includes the sample mean.

Statistical Signal Processing 1, Minor exam #1 Retake, 8-Jan-2024, 16:20-19:00

INSTRUCTIONS

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and let us collect the received samples to a vector

$$\mathbf{x} = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

Find observation matrix $\mathbf{H}$ and vector $\mathbf{w}$ such that $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$.

Solution: The observation matrix is:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cos(1) \\ 1 & 2 & \cos(2) \\ 1 & 3 & \cos(3) \\ 1 & 4 & \cos(4) \end{bmatrix}$$

and the the vector $\mathbf{w}$ is:

$$\mathbf{w} = \begin{bmatrix} w[1] \\ w[2] \\ w[3] \\ w[4] \end{bmatrix}$$

B. Problem 2

Assume that X_1 and X_2 are random variables with joint probability density function (PDF)

$$f(x_1, x_2) = \exp(-x_1 - x_2), 0 < x_1 < \infty, 0 < x_2 < \infty$$

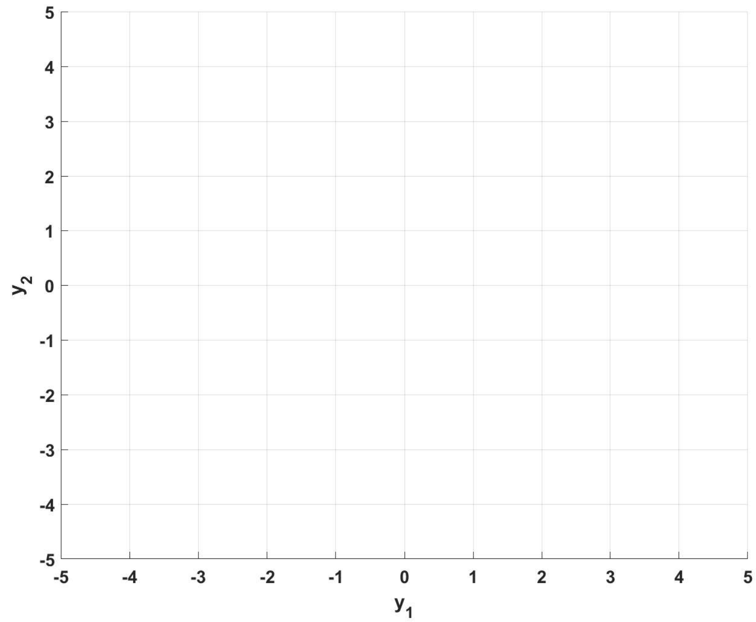
Let us consider transformation of random variables:

$$Y_1 = X_1 - X_2$$

and

$$Y_2 = X_1 + 2X_2$$

Find the joint PDF of Y_1 and Y_2. Make sure the find out the validity limits of the joint PDF. In addition to the equations, visually show the validity region of the joint PDF in the two dimensional xy-plane [draw a grid similar to the below figure]



Solution: We notice that

$$\begin{array}{l} X_2 = \frac{Y_2 - Y_1}{3} \\ X_1 = \frac{2}{3}Y_1 + \frac{1}{3}Y_2 \end{array}$$

Now, we know from the theory of transformation of random variables

$$f\left(y_1, y_2\right) = \exp\left(-\left(\frac{2}{3}Y_1 + \frac{1}{3}Y_2\right) - \left(\frac{Y_2 - Y_1}{3}\right)\right)|J| = \exp\left(-\left(\frac{1}{3}Y_1 + \frac{2}{3}Y_2\right)\right)|J|$$

where the Jacobian is

$$J = \det\left[\begin{array}{cc} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{array}\right] = \det\left[\begin{array}{cc} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{array}\right] = \frac{1}{3}$$

Finally, we get

$$f\left(y_1, y_2\right) = \frac{1}{3}\exp\left(-\left(\frac{1}{3}Y_1 + \frac{2}{3}Y_2\right)\right), 0 < \frac{2y_1 + y_2}{3} < \infty, 0 < \frac{y_2 - y_1}{3} < \infty$$

The limits can be equivalently written as

$$0 < 2y_1 + y_2 < \infty, 0 < y_2 - y_1 < \infty$$

from which we get

$$-\frac{y_2}{2} < y_1 < y_2, y_2 > 0$$

C. Problem 3

Assume that X is continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{5}{32}x^4 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

and further assume that

$$Y = X^2$$

- Find CDF of Y (find CDF for all values between $-\infty$ to ∞)
- Find PDF of Y.
- Find $E[Y]$ (give the solution as a rational number simplified as much as possible).

Solution: We notice that possible values of Y are between 0 and 4. Within this range,

$$P(Y < y) = P\left(X^2 < y\right) = P\left(0 \le X < \sqrt{y}\right) = \int_0^{\sqrt{y}} \frac{5}{32}x^4 dx = \frac{1}{32}y^{\frac{5}{2}}$$

For full solution, we need to include also points before and after the range

$$P(Y < y) = \begin{cases} 0 & y < 0 \\ \frac{1}{32}y^{\frac{5}{2}} & 0 \le y \le 4 \\ 1 & y > 4 \end{cases}$$

Now, PDF is easity to find using derivative

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{5}{64}y^{\frac{3}{2}} & 0 \le y \le 4 \\ 0 & y > 4 \end{cases}$$

And the expected value can be obtained with

$$E[Y] = \int_0^4 \frac{5}{64}y^{\frac{5}{2}}dy = \frac{5}{32}\frac{1}{7}2^7 = \frac{20}{7}$$

Statistical Signal Processing 1, Minor exam #2 Retake, 8-Jan-2024, 16:20-19:00

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A. *Problem 1*

The data

$$x[n] = 2A + w[n]$$

for $n = 0, 1, \cdots, N-1$ is observed where $w[n]$ is zero-mean white Gaussian noise with variance σ^2. We want to estimate unknown parameter A based on the observations $x[n]$.

1) Find the CRLB for A
2) Find the efficient estimator and verify that it is unbiased and reaches the CRLB.

Solution: By using the theory of signal in white Gaussian noise:

$$\text{CRLB} = \frac{\sigma^2}{\sum_{n=0}^{N-1}\left(\frac{\partial(2A)}{\partial A}\right)^2} = \frac{\sigma^2}{\sum_{n=0}^{N-1}(2)^2} = \frac{\sigma^2}{4N}$$

By using the standard approach:

$$p(\mathbf{X}) = \frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A)^2\right)$$
$$\log p(\mathbf{X}) = -\frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A)^2$$
$$\frac{\partial \log p(\mathbf{X})}{\partial A} = \frac{2}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A)$$
$$\frac{\partial^2 \log p(\mathbf{X})}{\partial A^2} = -\frac{4N}{\sigma^2}$$
$$\text{CRLB} = \frac{\sigma^2}{4N}$$

To get efficient estimator, we use the first derivative:

$$\frac{\partial \log p(\mathbf{X})}{\partial A} = \frac{2}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-2A) = \frac{2}{\sigma^2}\sum_{n=0}^{N-1}x[n] + \frac{2}{\sigma^2}\sum_{n=0}^{N-1}-2A$$
$$= \frac{2}{\sigma^2}\sum_{n=0}^{N-1}x[n] - \frac{4AN}{\sigma^2} = \frac{4N}{\sigma^2}\left(\frac{1}{2N}\sum_{n=0}^{N-1}x[n] - A\right)$$

Thefore, we get that the efficient estimator is

$$\widehat{A} = \frac{1}{2N}\sum_{n=0}^{N-1}x[n]$$

Let us check it is unbiased:

$$E\left[\widehat{A}\right] = \frac{1}{2N}\sum_{n=0}^{N-1} E\left[x[n]\right] = \frac{1}{2N}\sum_{n=0}^{N-1} 2A = \frac{2AN}{2N} = A$$

Let us check it reaches the CRLB:

$$\mathrm{Var}\left[\hat{A}\right] = \frac{1}{4N^2}\sum_{n=0}^{N-1} \mathrm{Var}\left[x[n]\right] = \frac{1}{4N^2}\sum_{n=0}^{N-1} \sigma^2 = \frac{\sigma^2 N}{4N^2} = \frac{\sigma^2}{4N}$$

B. Problem 2

Assume received data

$$x\left[n\right] = A\cos\left(2\pi f_1 n\right) + w[n]$$

for $n = 0, 1, \cdots, N-1$. The noise samples $w[n]$ are zero-mean and independent and identically distributed (IID) with variance σ^2. We want to to estimate unknown parameter A.

- Find BLUE estimator for A. Assume f_1 is known.
- What frequency f_1 (between 0 and 0.5) will give best results for this estimation problem?

1) Solution: The covariance matrix is $\mathbf{C} = \sigma^2\mathbf{I}$. Let us use the Gauss-Markov theorem (in simplified form since covariance matrix is just diagonal with all diagonal elements being the same). The observation matrix is

$$\mathbf{H} = \begin{bmatrix} \cos\left(0\right) \\ \cos\left(2\pi f_1\right) \\ \vdots \\ \cos\left(2\pi f_1\left(N-1\right)\right) \end{bmatrix}$$

Now we directly get the BLUE estimate as

$$\hat{A} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x} = \frac{\sum\limits_{n=0}^{N-1}\cos\left(2\pi f_1 n\right)x[n]}{\sum\limits_{n=0}^{N-1}\left[\cos\left(2\pi f_1 n\right)\right]^2}$$

The variance of the estimate is

$$Var\left(\hat{A}\right) = \sigma^2\left(\mathbf{H}^T\mathbf{H}\right)^{-1} = \frac{\sigma^2}{\sum\limits_{n=0}^{N-1}\left[\cos\left(2\pi f_1 n\right)\right]^2} \geq \frac{\sigma^2}{N}$$

The maximum value of cos is one (which is obtained when its argument is zero). Therefore, the best frequency to use is $f_1 = 0$.

C. Problem 3

Consider the observations following the model for observations $Y[k]$

$$Y\left[k\right] = \theta + w\left[k\right]$$

for $k = 0, 1, \cdots, N-1$. We want to estimate the unknown parameter θ. The zero-mean noise samples $w[k]$ are independent and have the PDF

$$p\left(x\right) = \frac{1}{2}\exp\left(-\left|x\right|\right)$$

for $-\infty < x < \infty$. Find all maximum likelihood estimates of θ when $N = 2$. Hint: Since noise is zero-mean, a very reasonable estimator (and valid ML estimate for $N = 2$) would be sample mean. You need to find all solutions in addition to the sample mean.

1) Solution: First we notice that the PDF of the observed samples is translation (shift) of the PDF of the noise

$$f\left(Y\left[k\right];\theta\right)=\frac{1}{2}\exp\left(-\left|Y\left[k\right]-\theta\right|\right)$$

Therefore, the joint PDF for $N=2$ is

$$f\left(\mathbf{Y};\theta\right)=\frac{1}{4}\exp\left(-\left|Y\left[0\right]-\theta\right|\right)\exp\left(-\left|Y\left[1\right]-\theta\right|\right)$$

Without loss of generality, let us sort the samples such that $Y[0]\leq Y[1]$. Now we have three possibilities for the location of θ. It can be less than $Y[0]$ (case 1), between $Y[0]$ and $Y[1]$ (case 2), or greater than $Y[1]$ (case 3). We now get the joint PDF for $N=2$ with

$$f\left(\mathbf{Y};\theta\right)=\begin{cases}\frac{1}{4}\exp\left(-Y[0]-Y[1]+2\theta\right) & \theta<Y\left[0\right]\\ \frac{1}{4}\exp\left(Y[0]-Y[1]\right) & Y\left[0\right]\leq\theta<Y\left[1\right]\\ \frac{1}{4}\exp\left(Y[0]+Y[1]-2\theta\right) & \theta\geq Y\left[1\right]\end{cases}$$

Let us do ML estimate for each case separately and then combine the results (choose to one that leads to maximum likelihood). For case 1, joint PDF is largest when θ is largest possible for that case

$$\hat{\theta}_1=Y\left[0\right]$$

The joint PDF at this θ is

$$\frac{1}{4}\exp\left(Y[0]-Y[1]\right)$$

For case 3, smallest θ gives the largest joint PDF so that

$$\hat{\theta}_3=Y\left[1\right]$$

For this case, the joint PDF at this θ

$$\frac{1}{4}\exp\left(Y[0]-Y[1]\right)$$

For case 2, any value between $Y[0]$ and $Y[1]$ is valid and the joint PDF is

$$\frac{1}{4}\exp\left(Y[0]-Y[1]\right)$$

We noticed that all above joint PDFs are equal. So any value between (including the end points) $Y[0]$ and $Y[1]$ can be chosen as the ML estimate. Obviously this includes the sample mean.

Statistical Signal Processing 1, Minor exam #3 Retake, 8-Jan-2024, 16:20-19:00

Instructions

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A. Problem 1

For the signal model

$$s[n] = \begin{cases} 2A & 0 \leq n \leq M-1 \\ -A & M \leq n \leq N-1 \end{cases}$$

derive the least squares estimator of A. Assume received noisy samples are $x[n], n = 0, 1, \cdots, N-1$.

Solution: The LSE minimizes

$$J = \sum_{n=0}^{N-1} (x[n] - s[n])^2 = \sum_{n=0}^{M-1} (x[n] - 2A)^2 + \sum_{n=M}^{N-1} (x[n] + A)^2$$

To minimize it, let us find the derivate with respect to A

$$\begin{aligned} \tfrac{\partial J}{\partial A} &= -4 \sum_{n=0}^{M-1} (x[n] - 2A) + 2 \sum_{n=M}^{N-1} (x[n] + A) \\ &= -4 \sum_{n=0}^{M-1} x[n] + 8AM + 2 \sum_{n=M}^{N-1} x[n] + 2A\,(N-M) \end{aligned}$$

Let us set the derivate to zero to find the LSE

$$\begin{aligned} -4 \sum_{n=0}^{M-1} x[n] + 8\hat{A}M + 2 \sum_{n=M}^{N-1} x[n] + 2\hat{A}\,(N-M) = 0 \\ \hat{A}\,(8M + 2\,(N-M)) = 4 \sum_{n=0}^{M-1} x[n] - 2 \sum_{n=M}^{N-1} x[n] \end{aligned}$$

Now, we can solve for $\hat{A}$ and get the LSE

$$\hat{A} = \frac{4 \sum_{n=0}^{M-1} x[n] - 2 \sum_{n=M}^{N-1} x[n]}{8M + 2\,(N-M)} = \frac{2 \sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n]}{4M + (N-M)} = \frac{2 \sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n]}{3M + N}$$

B. Problem 2

We want to estimate unknown random parameter θ. It has prior PDF

$$p(\theta) = \begin{cases} \exp(-\theta) & \theta > 0 \\ 0 & \theta \leq 0 \end{cases}$$

We have one observation

$$Y = \theta + w$$

where w is independent from θ and has the PDF

$$f(w) = \begin{cases} \exp(-w) & w > 0 \\ 0 & w \leq 0 \end{cases}$$

- Find the posterior PDF
- Find the MMSE estimator
- Find the MAP estimator

Solution: The posterior PDF is

$$p(\theta | Y) = \frac{p(Y | \theta) p(\theta)}{p(Y)}$$

We notice that Y is a translation (shifted by θ) of w. Therefore the conditional PDF is the same as the PDF of w but shifted by θ.

$$p(Y | \theta) = f(Y - \theta) = \begin{cases} \exp(-(Y - \theta)) & Y > \theta \\ 0 & Y \leq \theta \end{cases}$$

To get the unconditional PDF of Y we find

$$p(Y) = \int_0^Y p(Y | \theta) \exp(-\theta) \, d\theta = Y \exp(-Y)$$

Therefore, the posterior PDF is

$$p(\theta | Y) = \frac{p(Y | \theta) \exp(-\theta)}{p(Y)} = \begin{cases} \frac{1}{Y} & 0 < \theta < Y \\ 0 & \text{otherwise} \end{cases}$$

It has uniform distribution between 0 and Y. Therefore, the MMSE estimator (conditional mean of the posteriori PDF) is $\hat{\theta} = Y/2$. On on the other hand, the MAP is estimator is not unique and any value between 0 and Y can be chosen.

C. Problem 3

Assume the following PDF parameterized by a:

$$f(x; a) = \begin{cases} ax \exp\left(-\frac{ax^2}{2}\right) & x > 0, a > 0 \\ 0 & \text{otherwise} \end{cases}$$

This is a valid PDF which can be proven by integrating it by using the identity

$$\int ax \exp\left(-\frac{ax^2}{2}\right) dx = -\exp\left(-\frac{ax^2}{2}\right)$$

Assume two hypothesis H_0 and H_1 and one measurement x.

$$H_0 : f(x) = f(x; a_0)$$
$$H_1 : f(x) = f(x; a_1)$$

where $a_1 < a_0$. So for example under hypothesis H_0 the parameter $a = a_0$.

- Based on the measured value x, find the Neyman-Pearson detector (including deriving its threshold for the target probability of false alarm).
- Find the probability of detection as the function of the probability of false alarm.
- Assume (only for this subquestion) target probability of false alarm is $\alpha = 0.01$, $a_0 = 10$, and $a_1 = 5$. Find the numerical value of the obtained probability of detection (can be solved without calculator).

The problem is chosen in such way that we avoid the case where one hypothesis would be always chosen regardless of the value of x.

Solution: The Neyman-Pearson test is

$$\frac{a_1 x \exp\left(-\frac{a_1 x^2}{2}\right)}{a_0 x \exp\left(-\frac{a_0 x^2}{2}\right)} > \gamma$$

which can be simplified

$$\exp\left(\frac{a_0 - a_1}{2} x^2\right) > \frac{\gamma a_0}{a_1}$$

and finally we get

$$x^2 > \frac{2 \log\left(\frac{\gamma a_0}{a_1}\right)}{a_0 - a_1} = \gamma'$$

which can be written as (assuming right hand side is positive, which we can assume since it was said that one hypothesis is not always chosen and also since x is assumed always positive)

$$x > \gamma''$$

Assume that target probability of false alarm is α. Now we need that

$$P\left(x > \gamma''; H_0\right) = \int_{\gamma''}^{\infty} a_0 x \exp\left(-\frac{a_0 x^2}{2}\right) dx = \exp\left(-\frac{a_0 (\gamma'')^2}{2}\right) = \alpha$$

By solving this we get as the threshold

$$\gamma'' = \sqrt{\frac{-2\log(\alpha)}{a_0}}$$

The probability of detection is obtained as

$$P_D = P\left(x > \sqrt{\frac{-2\log(\alpha)}{a_0}}; H_1\right) = \int_{\sqrt{\frac{-2\log(\alpha)}{a_0}}}^{\infty} a_1 x \exp\left(-\frac{a_1 x^2}{2}\right) dx = \exp\left(\frac{a_1}{a_0}\log(\alpha)\right) = \alpha^{\left(\frac{a_1}{a_0}\right)}$$

For the last question: we directly use the above result to get that probability of detection is 0.1.