SSP1 Homeworks

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Abstract

The abstract goes here.

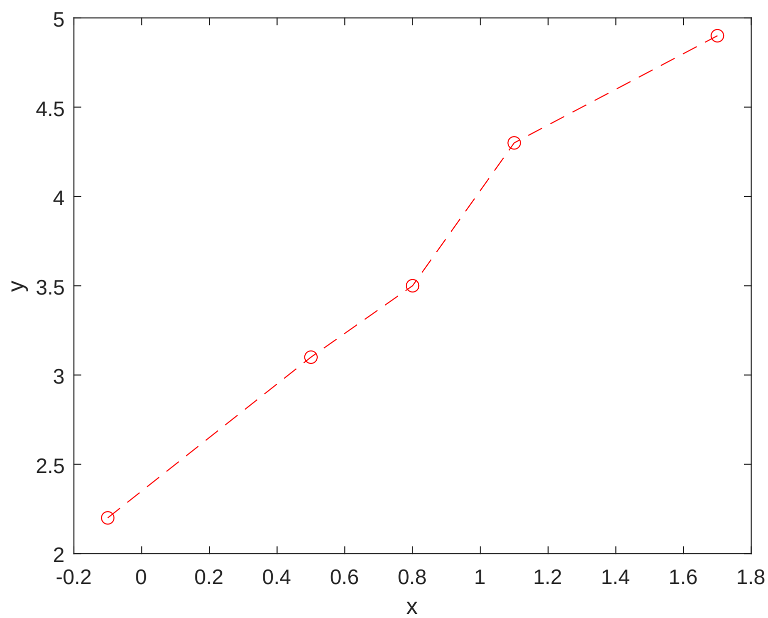
Index Terms

IEEE, IEEEtran, journal, LaTeX, paper, template.

I. Least squares estimation

A. Question 1

Assume that we have the following observed (x,y) pairs: (-0.1,2.2) (0.5, 3.1) (0.8,3.5) (1.1,4.3) (1.7,4.9). For example, the first observed point is at $x = -0.1$ with $y = 2.2$.



- Estimate parameters a and b of a line $y = ax + b$ with linear least squares estimation. First write the observation matrix $\mathbf{H}$ for this problem. You can solve the numerical matrix operations in MATLAB. Show also the estimated values for a and b in your answer.
- Estimate parameters a, b, c of a quadratic $y = ax + b + cx^2$ with linear least squares estimation. First write the observation matrix $\mathbf{H}$ for this problem. Show also the estimated values for a, b, c in your answer.

Answer: For the line, the observation matrix is

$$\mathbf{H} = \begin{bmatrix} -0.1 & 1 \\ 0.5 & 1 \\ 0.8 & 1 \\ 1.1 & 1 \\ 1.7 & 1 \end{bmatrix}$$

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and

$$\boldsymbol{\theta} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The solution (as the problem is linear least squares) is

$$\hat{\boldsymbol{\theta}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}$$

where

$$\mathbf{y} = \begin{bmatrix} 2.2 \\ 3.1 \\ 3.5 \\ 4.3 \\ 4.9 \end{bmatrix}$$

Numerically evaluating the solution, we get that $a = 1.55$ and $b = 2.36$.

For the second part, the observation matrix is

$$\mathbf{H} = \begin{bmatrix} -0.1 & 1 & (-0.1)^2 \\ 0.5 & 1 & (0.5)^2 \\ 0.8 & 1 & (0.8)^2 \\ 1.1 & 1 & (1.1)^2 \\ 1.7 & 1 & (1.7)^2 \end{bmatrix}$$

and

$$\boldsymbol{\theta} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

By using the linear least squares solution numerically, we get that $a = 1.6981$, $b = 2.3341$, $c = -0.0926$.

B. *Question 2*

Assume signal model

$$s[k] = k^3\theta_1 + 2k\theta_2 + 3\theta_2 + 1$$

for $k = 1, 2, 3$. Further assume that the noisy observations (assume zero-mean noise) are $x[1] = 7.1$, $x[2] = -25.3$, and $x[3] = -116.0$. Find the LSE and also show your estimated values for θ_1 and for θ_2. You can do the numerical matrix calculations in MATLAB.

Answer: Let us write the expression for the noisy observations

$$x[k] = s[k] + w[k]$$

which can be expanded to

$$x[k] = k^3\theta_1 + 2k\theta_2 + 3\theta_2 + 1 + w[k]$$

Now let us move the constant terms to left

$$x[k] - 1 = k^3\theta_1 + 2k\theta_2 + 3\theta_2 + w[k]$$

Now we get linear model

$$\tilde{x}[k] = \tilde{s}[k] + w[k]$$

where

$$\tilde{x}[k] = x[k] - 1$$

and

$$\tilde{s}[k] = k^3\theta_1 + 2k\theta_2 + 3\theta_2$$

The corresponding observation matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & 5 \\ 8 & 7 \\ 27 & 9 \end{bmatrix}$$

and the observation vector is

$$\tilde{\mathbf{x}} = \mathbf{x} - 1$$

Now, by using the theory is linear least squares the solution is

$$\hat{\theta} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\tilde{\mathbf{x}}$$

Numerically by MATLAB we find that

$$\begin{aligned} &\theta_1 \approx -5.0 \\ &\theta_2 \approx 2.1 \end{aligned}$$

C. *Question 3*

For the signal model

$$s[n] = \begin{cases} 2A & 0 \le n \le M-1 \\ -A & M \le n \le N-1 \end{cases}$$

derive the least squares estimator of A. Assume received noisy samples are $x[n], n = 0, 1, \cdots, N-1$.

Answer: The LSE minimizes

$$J = \sum_{n=0}^{N-1}\left(x[n] - s[n]\right)^2 = \sum_{n=0}^{M-1}\left(x[n] - 2A\right)^2 + \sum_{n=M}^{N-1}\left(x[n] + A\right)^2$$

To minimize it, let us find the derivate with respect to A

$$\begin{aligned} \tfrac{\partial J}{\partial A} &= -4\sum_{n=0}^{M-1}\left(x[n] - 2A\right) + 2\sum_{n=M}^{N-1}\left(x[n] + A\right) \\ &= -4\sum_{n=0}^{M-1}x[n] + 8AM + 2\sum_{n=M}^{N-1}x[n] + 2A\left(N-M\right) \end{aligned}$$

Let us set the derivate to zero to find the LSE

$$\begin{aligned} &-4\sum_{n=0}^{M-1}x[n] + 8\hat{A}M + 2\sum_{n=M}^{N-1}x[n] + 2\hat{A}\left(N-M\right) = 0 \\ &\hat{A}\left(8M + 2\left(N-M\right)\right) = 4\sum_{n=0}^{M-1}x[n] - 2\sum_{n=M}^{N-1}x[n] \end{aligned}$$

Now, we can solve for $\hat{A}$ and get the LSE

$$\hat{A} = \frac{4\sum_{n=0}^{M-1}x[n] - 2\sum_{n=M}^{N-1}x[n]}{8M + 2\left(N-M\right)} = \frac{2\sum_{n=0}^{M-1}x[n] - \sum_{n=M}^{N-1}x[n]}{4M + \left(N-M\right)} = \frac{2\sum_{n=0}^{M-1}x[n] - \sum_{n=M}^{N-1}x[n]}{3M+N}$$

D. *Question 4*

If the signal model is

$$s[n] = A + B(-1)^n$$

where $n = 0, 1, \cdots, N-1$ and N is even. Assume that the noisy observations are $x[n]$. Find the LSE of

$$\boldsymbol{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$$

Answer

Let us use the theory of linear least squares. For that we need find $\mathbf{H}$ for the below signal model

$$\mathbf{s} = \mathbf{H}\theta$$

We easily note that

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} = N\mathbf{I}$$

We know that the solution for the linear least squres is

$$\hat{\theta} = \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x} = \frac{\mathbf{H}^T\mathbf{x}}{N} = \begin{bmatrix} \frac{1}{N}\sum\limits_{n=0}^{N-1} x[n] \\ \frac{1}{N}\sum\limits_{n=0}^{N-1} (-1)^n x[n] \end{bmatrix}$$

II. Bayesian estimation

A. Question 1

Let us assume that the joint distribution of measurements z and θ is of the form

$$p\left(z, \theta\right) = e^{-\theta}$$

where $\theta > z, z > 0$. Find the MMSE estimator of θ based on the measurement z. Hint:

$$\int \theta e^{-\theta} d\theta = -e^{-\theta}\left(\theta + 1\right)$$

Answer: To get posterior PDF we need to first find $p(z)$. It is obtained as

$$p\left(z\right) = \int\limits_z^\infty p\left(z, \theta\right) d\theta = \int\limits_z^\infty e^{-\theta} d\theta = e^{-z}$$

The posteriori PDF is

$$p\left(\theta | z\right) = \frac{p\left(z, \theta\right)}{p\left(z\right)} = e^{-\left(\theta - z\right)}$$

where $\theta > z$. Now, the MMSE estimator is

$$\hat{\theta} = \int\limits_z^\infty \theta e^{-\left(\theta - z\right)} d\theta = e^z \int\limits_z^\infty \theta e^{-\theta} d\theta = e^z \left[e^{-z}\left(z + 1\right)\right] = z + 1$$

B. Question 2

We have measurements following the model $x[n] = A + w[n]$, where $n = 1, 2, \cdots, N$. The prior distribution for the unknown random parameter A is

$$p\left(A\right) = \begin{cases} \lambda \exp\left(-\lambda A\right) & A > 0 \\ 0 & \text{otherwise} \end{cases}$$

and the noise is white Gaussian noise with variance σ^2. We know σ^2 and λ. Find MAP estimate of the unknown random parameter A.

Answer: Let us store all the observations in a column vector $\mathbf{X}$. The conditional distribution of $\mathbf{X}$ given A is

$$p\left(\mathbf{X}|\, A\right) = \frac{1}{\left(2\pi\sigma^2\right)^{N/2}} \exp\left(-\frac{\sum\limits_{n=1}^{N}\left(x[n]-A\right)^2}{2\sigma^2}\right)$$

Now the joint distribution is

$$p\left(\mathbf{X}, A\right) = \begin{cases} \frac{\lambda \exp(-\lambda A)}{\left(2\pi\sigma^2\right)^{N/2}} \exp\left(-\frac{\sum\limits_{n=1}^{N}\left(x[n]-A\right)^2}{2\sigma^2}\right) & A > 0 \\ 0 & \text{otherwise} \end{cases}$$

In order to maximize it, let us assume that $A > 0$ and take the logarithm.

$$\log p\left(\mathbf{X}, A\right) = \log\left(\lambda\right) - \lambda A - \frac{N}{2}\log\left(2\pi\sigma^2\right) - \frac{\sum\limits_{n=1}^{N}\left(x[n]-A\right)^2}{2\sigma^2}$$

The derivative with respect to A is

$$\frac{\partial \log p\left(\mathbf{X}, A\right)}{\partial A} = -\lambda + \frac{\sum\limits_{n=1}^{N}\left(x[n]-A\right)}{\sigma^2}$$

We set the derivate to zero and solve for A and get

$$\hat{A} = \frac{\sum\limits_{n=1}^{N} x[n]}{N} - \frac{\lambda\sigma^2}{N}$$

However, we need to check that estimate is always greater than zero. The second derivative is always negative. Therefore, the estimate is

$$\hat{A}_{\text{MAP}} = \max\left(0, \frac{\sum\limits_{n=1}^{N} x[n]}{N} - \frac{\lambda\sigma^2}{N}\right)$$

C. Question 3

We observe the data $x[n]$ for $n = 0, 1, \cdots, N-1$, where $x[n]$ has the conditional PDF (samples are conditinally independent)

$$p\left(x[n]|\, \lambda\right) = \frac{\lambda}{2}\exp\left(-\lambda\left|x[n]\right|\right)$$

the prior distribution $p(\lambda)$ is given by

$$p\left(\lambda\right) = \begin{cases} 1/\alpha & c \le \lambda \le \alpha + c \\ 0 & \text{otherwise} \end{cases}$$

Determine (in as simplified form as possible) the MAP estimator for λ using N samples for the cases (1) $c > 0$ and (2) $c = 0$. Hint: remember to consider the validity range of λ.

Answer: First, we find the joint conditional PDF

$$p\left(\mathbf{X}|\,\lambda\right) = \frac{\lambda^N}{2^N}\exp\left(-\lambda\sum_{n=0}^{N-1}|x[n]|\right)$$

The MAP estimator is given by

$$\hat{\lambda} = \arg\max_{\lambda} p\left(\mathbf{X}|\,\lambda\right)p\left(\lambda\right)$$

where

$$p\left(\mathbf{X}|\,\lambda\right)p\left(\lambda\right) = \begin{cases} \frac{\lambda^N}{\alpha 2^N}\exp\left(-\lambda\sum\limits_{n=0}^{N-1}|x[n]|\right) & c \le \lambda \le \alpha + c \\ 0 & \text{otherwise} \end{cases}$$

The maximum value must occur in the range $c \le \lambda \le \alpha + c$. Let us assume that and find

$$\log\left(p\left(\mathbf{X}|\,\lambda\right)p\left(\lambda\right)\right) = N\log\left(\lambda\right) - N\log(2) - \log\left(\alpha\right) - \lambda\sum_{n=0}^{N-1}|x[n]|$$

which is

$$\frac{\partial\log\left(p\left(\mathbf{X}|\,\lambda\right)p\left(\lambda\right)\right)}{\partial\lambda} = \frac{N}{\lambda} - \sum_{n=0}^{N-1}|x[n]|$$

To find the maximum value let us set the derivative to zero and solve for $\hat{\theta}$. We get

$$\hat{\lambda} = \frac{1}{\frac{1}{N}\sum\limits_{n=0}^{N-1}|x[n]|}$$

But we have to make sure that this is in the valid range. If the value if greater than $\alpha + c$, we know that maximum in the valid range occurs at $\alpha + c$ (since the second derivative is always negative). If the value is less than c, the maximum occurs at c. Therefore, we get

$$\hat{\lambda} = \max\left(c, \min\left(\alpha + c, \frac{1}{\frac{1}{N}\sum\limits_{n=0}^{N-1}|x[n]|}\right)\right)$$

When $c = 0$, we get

$$\hat{\lambda} = \max\left(0, \min\left(\alpha, \frac{1}{\frac{1}{N}\sum\limits_{n=0}^{N-1}|x[n]|}\right)\right)$$

but we notice that

$$\frac{1}{\frac{1}{N}\sum\limits_{n=0}^{N-1}|x[n]|}$$

is always positive. Therefore, the max-operation is not needed and the answer is

$$\hat{\lambda} = \min\left(\alpha, \frac{1}{\frac{1}{N}\sum\limits_{n=0}^{N-1}|x[n]|}\right)$$

III. Statistical decision theory and detection of signals

A. Question 1

Determine under what conditions perfect detector ($\mathrm{P}_{\mathrm{FA}} = 0$ and $\mathrm{P}_{\mathrm{D}} = 1$) for the problem

$$H_0 : x[0] \sim \; U[-c, c]$$
$$H_1 : x[0] \sim \; U[1-c, 1+c]$$

where $c > 0$ and $U[a, b]$ denotes a uniform PDF on the interval $[a, b]$, by choosing c.

Answer: We notice that perfect detection is possible when $1 - c > c$. Therefore, detection will be perfect when

$$0 < c < \frac{1}{2}$$

B. Question 2

Find the minimum error probability / MAP decision rule for detecting whether H_1 or H_0 is true based on one sample x that follows

$$H_0 : x \sim N(0, 1)$$
$$H_1 : x \sim N(0, 3)$$

if $P(H_0) = 2/3$ and also if $P(H_0) = 1/3$.

Answer: We know that the MAP decision rule is

$$\frac{p\left(x \middle| H_1\right)}{p\left(x \middle| H_0\right)} > \frac{P\left(H_0\right)}{P\left(H_1\right)}$$

First, we notice that $P(H_1) = 1 - P(H_0)$. The Gaussian PDF is

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

where σ^2 is the variance and μ is the mean. Therefore, we get as the MAP decision rule

$$\frac{\frac{1}{\sqrt{2\pi 3}} \exp\left(-\frac{1}{6}x^2\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)} > \frac{P\left(H_0\right)}{1 - P\left(H_0\right)}$$

which can be simplified to

$$\frac{1}{\sqrt{3}} \exp\left(\frac{1}{3}x^2\right) > \frac{P\left(H_0\right)}{1 - P\left(H_0\right)}$$

which can be solved for x^2

$$x^2 > 3 \log\left(\sqrt{3}\left(\frac{P\left(H_0\right)}{1 - P\left(H_0\right)}\right)\right)$$

Taking square root we get

$$|x| > \sqrt{3 \log\left(\sqrt{3}\left(\frac{P\left(H_0\right)}{1 - P\left(H_0\right)}\right)\right)}$$

Detector will choose H_1 if this condition is true. Now for $P(H_0) = 2/3$ we get

$$|x| > \sqrt{3 \log\left(2\sqrt{3}\right)} \approx 1.9306$$

For $P(H_0) = 1/3$ we notice that the threshold for x^2 is negative

$$x^2 > 3 \log\left(\frac{\sqrt{3}}{2}\right) \approx -0.4315$$

Therefore, this condition is always true. Detector will always choose H_1.

C. Question 3

Determine the number of samples N required for DC level A in the WGN detection problem so that $\mathrm{P_{FA}} = 10^{-2}$ and $\mathrm{P_D} = 0.99$. We know that the SNR is $10\log_{10}(A^2/\sigma^2) = -30$ dB. Assume that $A > 0$.

Answer: We know that NP decision rule is

$$\frac{p\left(x|\,H_1\right)}{p\left(x|\,H_0\right)} > \gamma$$

The Gaussian PDF for a single sample is

$$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Since the samples are independent we get as the NP test

$$\frac{\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right)}{\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2\right)} > \gamma$$

which can be simplified to

$$A\sum_{n=0}^{N-1}x[n] > \sigma^2\log(\gamma) + \frac{1}{2}\sum_{n=0}^{N-1}A^2$$

Now we get

$$\frac{1}{N}\sum_{n=0}^{N-1}x[n] > \frac{\sigma^2}{AN}\log(\gamma) + \frac{A}{2} = \gamma'$$

The test statistic is linear combination of independent Gaussians. Therefore, it is also Gaussian. The mean under H_0 is 0 and variance under H_0 is σ^2/N. Therefore, the probability of false alarm is

$$P_{\mathrm{FA}} = Q\left(\frac{\gamma'}{\sqrt{\frac{\sigma^2}{N}}}\right)$$

Now, the threshold for given probability of false alarm is

$$\gamma' = \sqrt{\frac{\sigma^2}{N}}Q^{-1}\left(P_{\mathrm{FA}}\right)$$

To get probability of detection, we need to find mean and variance under H_1. Variance remains the same but now the mean is A. Therefore, probability of detection is

$$P_{\mathrm{D}} = Q\left(\frac{\gamma'-A}{\sqrt{\frac{\sigma^2}{N}}}\right) = Q\left(Q^{-1}\left(P_{\mathrm{FA}}\right) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$

Let us solve for N:

$$N = \frac{\sigma^2}{A^2}\left(Q^{-1}\left(P_{\mathrm{FA}}\right) - Q^{-1}\left(P_{\mathrm{D}}\right)\right)^2$$

We know that

$$10\log_{10}\left(\frac{A^2}{\sigma^2}\right) = -30$$

Therefore,

$$\frac{A^2}{\sigma^2} = 0.001$$

Finally, we get the required number of samples N as (after rounding up)

$$N = 21648$$

REFERENCES

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.