SSP1 Homeworks (Minor Exam #2)

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Abstract

The abstract goes here.

Index Terms

IEEE, IEEEtran, journal, LaTeX, paper, template.

I. MVU Estimator

A. Question 1

We are using estimator

$$\widehat{\theta} = \frac{1}{aN} \sum_{i=1}^{N} \left(z[i]^2 + b \right)$$

where a and b are constants. In addition we know that $E\left[z[i]^2 \right] = 3\left(\theta - 2 \right)$. Determine a and b so that $\widehat{\theta}$ is unbiased estimator of θ.

Answer:

$$E\left[\widehat{\theta} \right] = \frac{1}{aN} \sum_{i=1}^{N} E\left[z[i]^2 + b \right] = \frac{1}{aN} \sum_{i=1}^{N} \left(3\left(\theta - 2 \right) + b \right)$$
$$= \frac{1}{aN} \sum_{i=1}^{N} \left(3\theta - 6 + b \right) = \frac{3\theta - 6 + b}{a}$$

It is obvious that the solution is $a = 3$ and $b = 6$.

B. Question 2

Suppose we have random observations given by

$$Y\left[k\right] = \theta_1 + W[k],\ k = 0, \cdots, N-1$$

where each $W[k]$ is independent and identically distributed Gaussian random variable with mean 0 and variance θ_2, so that $W[k] \sim N\left(0, \theta_2\right)$. Note that both θ_1 and $\theta_2 > 0$ are unknown. This is called a vector estimation problem.

We known an unbiased estimator for θ_1: the sample mean $\hat{\theta}_1 = \frac{1}{N} \sum_{k=0}^{N-1} Y[k]$. This estimator is still valid in this case because the sample mean does not depend on any unknown parameters.

How about this estimator for θ_2:

$$\hat{\theta}_2 = \frac{1}{N} \sum_{k=0}^{N-1} \left(Y\left[k\right] - \hat{\theta}_1 \right)^2$$

Is this estimator unbiased?

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Answer:

$$
\begin{aligned}
&E\left[\hat{\theta}_2\right] = \tfrac{1}{N}\sum_{l=0}^{N-1} E\left[\left(Y[l]-\hat{\theta}_1\right)^2\right]\\
&= \tfrac{1}{N}\sum_{l=0}^{N-1} E\left[\left(Y[l]-\tfrac{1}{N}\sum_{k=0}^{N-1} Y[k]\right)^2\right]\\
&= \tfrac{1}{N}\sum_{l=0}^{N-1} E\left[\left(\theta_1+W[l]-\tfrac{1}{N}\sum_{k=0}^{N-1}(\theta_1+W[k])\right)^2\right]\\
&= \tfrac{1}{N}\sum_{l=0}^{N-1} E\left[\left(W[l]-\tfrac{1}{N}\sum_{k=0}^{N-1} W[k]\right)^2\right]\\
&= \tfrac{1}{N}\sum_{l=0}^{N-1} E\left[\left(\tfrac{N}{N}W[l]-\tfrac{1}{N}\sum_{k=0}^{N-1} W[k]\right)^2\right]\\
&= \tfrac{1}{N^3}\sum_{l=0}^{N-1} E\left[\left(NW[l]-\sum_{k=0}^{N-1} W[k]\right)^2\right]\\
&= \tfrac{1}{N^3}\sum_{l=0}^{N-1} E\left[\left((N-1)W[l]-\sum_{k=0,k\neq l}^{N-1} W[k]\right)^2\right]\\
&= \tfrac{1}{N^3}\sum_{l=0}^{N-1} E\left[\left((N-1)W[l]-\sum_{k=0,k\neq l}^{N-1} W[k]\right)\left((N-1)W[l]-\sum_{j=0,j\neq l}^{N-1} W[j]\right)\right]\\
&= \tfrac{1}{N^3}\sum_{l=0}^{N-1}\left((N-1)^2\right)E\left[W[l]^2\right]+0+0+\tfrac{1}{N^3}\sum_{l=0}^{N-1}\left[\sum_{k=0,k\neq l}^{N-1}\sum_{j=0,j\neq l}^{N-1}E\left[W[k]W[j]\right]\right]\\
&= \tfrac{1}{N^3}\sum_{l=0}^{N-1}\left((N-1)^2\right)E\left[W[l]^2\right]+\tfrac{1}{N^3}\sum_{l=0}^{N-1}\left[\sum_{k=0,k\neq l}^{N-1}E\left[W[k]^2\right]\right]\\
&= \tfrac{1}{N^3}\sum_{l=0}^{N-1}\left((N-1)^2\right)\theta_2+\tfrac{1}{N^3}\sum_{l=0}^{N-1}\left[(N-1)\,\theta_2\right]=\tfrac{\theta_2}{N^2}\left(\left((N-1)^2\right)+(N-1)\right)\\
&= \tfrac{\theta_2}{N^2}\left(N^2-2N+1+(N-1)\right)=\tfrac{\theta_2}{N^2}\left(N^2-N\right)=\theta_2\left(1-\tfrac{1}{N}\right)=\theta_2\left(\tfrac{N-1}{N}\right)
\end{aligned}
$$

Therefore, the estimator is biased. Note that we can ”correct” the bias by using instead:

$$\hat{\theta}_2 = \frac{1}{N-1}\sum_{k=0}^{N-1}\left(Y[k]-\hat{\theta}_1\right)^2$$

C. Question 3

Find an unbiased estimator for the unknown scalar parameter θ given independent and identically distributed observations $Y[k]$, $k=0,\cdots,N-1$. Each observation follows the uniform distribution between $-\theta$ and θ, i.e., $Y\left[k\right]\sim U\left(-\theta,\theta\right)$.

Answer: Let us make a new random variable $Z[k]=|Y[k]|$. Note that $Z\left[k\right]\sim U\left(0,\theta\right)$. Therefore,

$$E\left[\frac{1}{N}\sum_{k=0}^{N-1} Z[k]\right]=\frac{\theta}{2}$$

Then an unbiased estimator for θ would be

$$\hat{\theta}=\frac{2}{N}\sum_{k=0}^{N-1}|Y[k]|$$

We can confirm that

$$E\left[\hat{\theta}\right]=\frac{2}{N}\sum_{k=0}^{N-1}E\left|Y[k]\right|=\frac{2}{N}\sum_{k=0}^{N-1}\frac{\theta}{2}=\frac{2}{N}\frac{\theta}{2}N=\theta$$

D. Question 4

Suppose you have an unknown scalar parameter θ and get two independent and identically distributed observations $Y[0]$ and $Y[1]$ with the observation model

$$Y\left[k\right]\sim U\left(0,\theta\right)$$

for $k=0$ and $k=1$. Consider the following two estimators:

$$\begin{array}{l}\hat{\theta}_a=Y\left[0\right]+Y\left[1\right]\\ \hat{\theta}_b=\frac{3}{2}\max\left(Y\left[0\right],Y\left[1\right]\right)\end{array}$$

where the function max outputs the larger of the two inputs. Are both estimators unbiased? Hint: the distribution of $Z=\max\left(Y\left[0\right],Y\left[1\right]\right)$ is

$$f_Z\left(z\right)=\left\{\begin{array}{cc}\frac{2z}{\theta^2} & 0\le z\le\theta\\ 0 & \text{otherwise}\end{array}\right.$$

Which estimator is better? Explain.

Answer:

$$E\left[\hat{\theta}_a\right]=E\left[Y\left[0\right]\right]+E\left[Y\left[1\right]\right]=\frac{\theta}{2}+\frac{\theta}{2}=\theta$$

$$\begin{array}{l}E\left[\hat{\theta}_b\right]=\frac{3}{2}E\left[\max\left(Y\left[0\right],Y\left[1\right]\right)\right]\\ =\frac{3}{2}\int\limits_0^\theta\frac{2z}{\theta^2}zdz=\frac{3}{2}\int\limits_0^\theta\frac{2z^2}{\theta^2}dz=\frac{3}{\theta^2}\int\limits_0^\theta z^2dz\\ =\frac{3}{\theta^2}\left(\frac{\theta^3}{3}-0\right)=\theta\end{array}$$

So both estimators are unbiased, to find out which is better we need to compute variances (as both are unbiased).

$$\text{Var}\left[\hat{\theta}_a\right]=\text{Var}\left[Y\left[0\right]\right]+\text{Var}\left[Y\left[1\right]\right]=\frac{\theta^2}{12}+\frac{\theta^2}{12}=\frac{\theta^2}{6}$$

$$\begin{array}{l}\text{Var}\left[\hat{\theta}_b\right]=\frac{9}{4}\text{Var}\left[\max\left(Y\left[0\right],Y\left[1\right]\right)\right]\\ =\frac{9}{4}\left(\int\limits_0^\theta\frac{2z}{\theta^2}\left(z-\frac{2}{3}\theta\right)^2dz\right)=\frac{9}{2\theta^2}\left(\int\limits_0^\theta\left(z^3-\frac{4}{3}\theta z^2+\frac{4}{9}\theta^2z\right)dz\right)\\ =\frac{9}{2\theta^2}\left(\frac{\theta^4}{4}-\frac{4\theta^4}{9}+\frac{2\theta^4}{9}\right)=\frac{\theta^2}{8}\end{array}$$

Therefore, estimator $\hat{\theta}_b$ is better since it has lower variance.

E. Question 5

Consider the data $X[k],\ \ k=0,1,\cdots,N-1$, where each sample is independent and identically distributed as $U\left(0,\theta\right)$ (uniform distribution between 0 and θ). Find unbiased estimator for θ.

Answer: Mean of each sample is

$$E\left[X[k]\right] = \int_0^\theta \frac{1}{\theta} z dz = \frac{1}{\theta}\frac{\theta^2}{2} = \frac{\theta}{2}$$

Therefore, we can easily note that unbiased estimator is

$$\hat{\theta} = \frac{2}{N}\sum_{k=0}^{N-1} X[k]$$

We can confirm this by

$$\begin{array}{l} E\left[\hat{\theta}\right] = \frac{2}{N}\sum\limits_{k=0}^{N-1} E\left[X[k]\right] = \frac{2}{N}\sum\limits_{k=0}^{N-1}\frac{\theta}{2} \\ = \frac{2}{N}\frac{\theta N}{2} = \theta \end{array}$$

F. *Question 6*

The data

$$X[k],\ k = 0, 1$$

are observed where each sample is independent and identically distributed as $N\left(0, \sigma^2\right)$. We wish to estimate the variance σ^2 with

$$\widehat{\sigma^2} = \frac{1}{2}\left[X[0]^2 + X[1]^2\right]$$

The estimator is unbiased. Find its probability density function. Hint: study about chi-squared distribution and multiplication of a random variable with a constant.

Answer:

$$\begin{array}{l} \widehat{\sigma^2} = \frac{\sigma^2}{2}\left[\frac{X[0]^2}{\sigma^2} + \frac{X[1]^2}{\sigma^2}\right] \sim \frac{\sigma^2}{2}\chi^2\left(2\right) \\ f\left(\chi^2\left(2\right)\right) = \frac{1}{2}e^{-x/2}, x > 0 \\ f\left(\widehat{\sigma^2}\right) = \left\{ \begin{array}{cc} \frac{2}{\sigma^2}\frac{1}{2}e^{-\frac{\widehat{\sigma^2}}{2}\frac{2}{\sigma^2}} = \frac{1}{\sigma^2}e^{-\frac{\widehat{\sigma^2}}{\sigma^2}} & \widehat{\sigma^2} > 0 \\ 0 & \text{otherwise} \end{array} \right. \end{array}$$

G. *Question 7*

Consider the observations

$$X\left[n\right] = A + W[n],\ n = 0, 1, \cdots, N-1$$

where A is the unknown parameter to be estimated and $W[n]$ is additive white gaussian noise with variance σ^2. A reasonable unbiased estimator for A is the sample mean:

$$\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} X\left[n\right]$$

Suppose we want to estimate $\theta = A^2$. Is

$$\hat{\theta} = \left(\frac{1}{N}\sum_{n=0}^{N-1} X\left[n\right]\right)^2$$

unbiased?

Answer: We remember that

$$\mathrm{Var}\left[X\left[n\right]\right] = E\left[X[n]^2\right] - E[X\left[n\right]]^2$$
$$E\left[X[n]^2\right] = \mathrm{Var}\left[X\left[n\right]\right] + E[X\left[n\right]]^2$$

Therefore,

$$E\left[X[n]^2\right] = \sigma^2 + A^2$$

And when $k \neq n$

$$E\left[X\left[n\right]X[k]\right] = E\left[\left(A + W[n]\right)\left(A + W[k]\right)\right]$$
$$= E\left[A^2 + AW[k] + W[n]A + W[n]W[k]\right] = A^2$$

We get that

$$E\left[\hat{\theta}\right] = \tfrac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} E\left[X\left[n\right]X\left[k\right]\right] = \tfrac{1}{N^2}\left(\sum_{n=0}^{N-1} E\left[X[n]^2\right] + N\left(N-1\right)A^2\right)$$
$$\tfrac{1}{N^2}\left(\left(\sigma^2 + A^2\right)N + N\left(N-1\right)A^2\right) = A^2 + \tfrac{\sigma^2}{N}$$

Also, easier way,

$$E\left[\hat{\theta}\right] = E\left[\hat{A}^2\right] = \mathrm{Var}\left[\hat{A}\right] + E\left[\hat{A}\right]^2 = \frac{\sigma^2}{N} + A^2$$

As $N \to \infty$, the estimator $\hat{\theta}$ becomes (asymptotically) unbiased.

H. *Question 8*

The data

$$X[k],\ k = 0, 1, \cdots, N-1$$

are observed where each sample is independent and identically distributed as $N\left(0, \sigma^2\right)$. We wish to estimate the variance σ^2 with

$$\widehat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} X[n]^2$$

Notice that mean is known to be zero (it does not need to be estimated). Is this an unbiased estimator? Also find the variance of $\widehat{\sigma^2}$.

Answer: Let us remember that

$$\mathrm{Var}\left[X\left[n\right]\right] = E\left[X[n]^2\right] - E[X\left[n\right]]^2$$
$$E\left[X[n]^2\right] = \mathrm{Var}\left[X\left[n\right]\right] + E[X\left[n\right]]^2$$

Therefore,

$$E\left[X[n]^2\right] = \sigma^2$$

and we get

$$E\left[\widehat{\sigma^2}\right] = \frac{1}{N}\sum_{n=0}^{N-1} E\left[X[n]^2\right] = \frac{1}{N}\sum_{n=0}^{N-1}\sigma^2 = \sigma^2$$

Therefore, the estimator is unbiased. For the variance, we first note by using properties of moments of Gaussian random variables (with zero-mean as here):

$$\mathrm{Var}\left[X[n]^2\right] = E\left[X[n]^4\right] - E\left[X[n]^2\right]^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4$$

$$\mathrm{Var}\left[\widehat{\sigma^2}\right] = \frac{1}{N^2}\sum_{n=0}^{N-1}\mathrm{Var}\left[X[n]^2\right] = \frac{1}{N^2}\sum_{n=0}^{N-1} 2\sigma^4 = \frac{2\sigma^4 N}{N^2} = \frac{2\sigma^4}{N}$$

II. CRLB

A. *Question 1*

The data

$$x[n] = 2A + w[n]$$

for $n = 0, 1, \cdots, N-1$ is observed where $w[n]$ is zero-mean white Gaussian noise with variance σ^2. We want to estimate unknown parameter A based on the observations $x[n]$.

1) Find the CRLB for A
2) Find the efficient estimator and verity that it is unbiased and reaches the CRLB.

Answer: By using the theory of signal in white Gaussian noise:

$$\text{CRLB} = \frac{\sigma^2}{\sum\limits_{n=0}^{N-1} \left(\frac{\partial(2A)}{\partial A}\right)^2} = \frac{\sigma^2}{\sum\limits_{n=0}^{N-1} (2)^2} = \frac{\sigma^2}{4N}$$

By using the standard approach:

$$\begin{aligned}
&p(\mathbf{X}) = \tfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\tfrac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - 2A)^2\right) \\
&\log p(\mathbf{X}) = -\tfrac{N}{2} \log(2\pi\sigma^2) - \tfrac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - 2A)^2 \\
&\tfrac{\partial \log p(\mathbf{X})}{\partial A} = \tfrac{2}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - 2A) \\
&\tfrac{\partial^2 \log p(\mathbf{X})}{\partial A^2} = -\tfrac{4N}{\sigma^2} \\
&\text{CRLB} = \tfrac{\sigma^2}{4N}
\end{aligned}$$

To get efficient estimator, we use the first derivative:

$$\begin{aligned}
&\tfrac{\partial \log p(\mathbf{X})}{\partial A} = \tfrac{2}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - 2A) = \tfrac{2}{\sigma^2} \sum_{n=0}^{N-1} x[n] + \tfrac{2}{\sigma^2} \sum_{n=0}^{N-1} -2A \\
&= \tfrac{2}{\sigma^2} \sum_{n=0}^{N-1} x[n] - \tfrac{4AN}{\sigma^2} = \tfrac{4N}{\sigma^2} \left(\tfrac{1}{2N} \sum_{n=0}^{N-1} x[n] - A\right)
\end{aligned}$$

Thefore, we get that the efficient estimator is

$$\widehat{A} = \frac{1}{2N} \sum_{n=0}^{N-1} x[n]$$

Let us check it is unbiased:

$$E\left[\widehat{A}\right] = \frac{1}{2N} \sum_{n=0}^{N-1} E[x[n]] = \frac{1}{2N} \sum_{n=0}^{N-1} 2A = \frac{2AN}{2N} = A$$

Let us check it reaches the CRLB:

$$\text{Var}\left[\hat{A}\right] = \frac{1}{4N^2} \sum_{n=0}^{N-1} \text{Var}[x[n]] = \frac{1}{4N^2} \sum_{n=0}^{N-1} \sigma^2 = \frac{\sigma^2 N}{4N^2} = \frac{\sigma^2}{4N}$$

B. *Question 2*

Radar measurements typically obey the Rayleigh-distribution

$$p\left(Z\left[i\right];\alpha\right)=\frac{Z\left[i\right]}{\alpha^{2}}\exp\left(\frac{-Z[i]^{2}}{2\alpha^{2}}\right)$$

where $Z[i]>0$. Let us assume that we are given N statistically independent measurements $Z[1],Z[2],\cdots,Z[N]$. From the Cramer-Rao lower bound for an estimator of α. We know that

$$E\left[Z[i]^{2}\right]=2\alpha^{2}$$

Answer:

$$p\left(\mathbf{Z}\right)=\frac{\prod\limits_{i=1}^{N}Z[i]}{\alpha^{2N}}\exp\left(\frac{-\sum\limits_{i=1}^{N}Z[i]^{2}}{2\alpha^{2}}\right)$$

$$\log p\left(\mathbf{Z}\right)=\sum_{i=1}^{N}\log\left(Z\left[i\right]\right)-2N\log\left(\alpha\right)-\frac{\sum\limits_{i=1}^{N}Z[i]^{2}}{2\alpha^{2}}$$

$$\frac{\partial\log p(\mathbf{Z})}{\partial\alpha}=-\frac{2N}{\alpha}+\frac{\sum\limits_{i=1}^{N}Z[i]^{2}}{\alpha^{3}}$$

$$\frac{\partial^{2}\log p(\mathbf{Z})}{\partial\alpha^{2}}=\frac{2N}{\alpha^{2}}-3\frac{\sum\limits_{i=1}^{N}Z[i]^{2}}{\alpha^{4}}$$

$$E\left[\frac{\partial\log p(\mathbf{Z})}{\partial\alpha^{2}}\right]=\frac{2N}{\alpha^{2}}-3\frac{\sum\limits_{i=1}^{N}E\left[Z[i]^{2}\right]}{\alpha^{4}}=\frac{2N}{\alpha^{2}}-6\frac{\alpha^{2}N}{\alpha^{4}}=-\frac{4N}{\alpha^{2}}$$

$$\text{CRLB}=-\frac{1}{E\left[\frac{\partial\log p(\mathbf{Z})}{\partial\alpha^{2}}\right]}=\frac{\alpha^{2}}{4N}$$

C. *Question 3*

If the data $X[n]$, for $n=0,1,\cdots,N-1$ are independent and identically distributed with uniform distribution $U[0,\theta]$, show that the regularity condition does not hold such as:

$$E\left[\frac{\partial\log p\left(\mathbf{X};\theta\right)}{\partial\theta}\right]\neq 0,\ \text{for }\theta>0$$

Hence the CRLB can not be applied to this problem.

Answer:

$$p\left(\mathbf{X};\theta\right)=\begin{cases}\frac{1}{\theta^{N}} & 0\le X[n]\le\theta,\forall n\\ 0 & \text{otherwise}\end{cases}$$

$$\log p\left(\mathbf{X};\theta\right)=-N\log\theta$$

$$\frac{\partial\log p(\mathbf{X};\theta)}{\partial\theta}=\frac{-N}{\theta}$$

$$E\left[\frac{\partial\log p(\mathbf{X};\theta)}{\partial\theta}\right]=\frac{-N}{\theta}\neq 0$$

D. *Question 4*

The data

$$x[n]=Ar^{n+4}+\ w[n]$$

for $n=0,1,2,...N-1$ are observed, where $w[n]$ is WGN with variance σ^{2} and $r>0$ is known.

1) Write the log-likelihood function for the observed vector X.
2) Derive the CRLB for the parameter A by using the theory of signals in WGN.
3) Derive the MVU estimator for A.

Answer: (a)

$$p\left(X\right)=\frac{1}{\left(2\pi\sigma^2\right)^{N/2}}\exp\left(\frac{-\sum\limits_{n=0}^{N-1}\left(x[n]-Ar^{n+4}\right)^2}{2\sigma^2}\right)$$

$$\log p\left(X\right)=-\frac{N}{2}\log\left(2\pi\sigma^2\right)-\frac{\sum\limits_{n=0}^{N-1}\left(x[n]-Ar^{n+4}\right)^2}{2\sigma^2}$$

(b)

$$\text{CRLB}=\frac{\sigma^2}{\sum\limits_{n=0}^{N-1}\left(\frac{\partial Ar^{n+4}}{\partial A}\right)^2}=\frac{\sigma^2}{\sum\limits_{n=0}^{N-1}\left(r^{n+4}\right)^2}=\frac{\sigma^2}{\sum\limits_{n=0}^{N-1}r^{2(n+4)}}$$

Using standard approach we also get the same answer:

$$\log p\left(X\right)=-\frac{N}{2}\log\left(2\pi\sigma^2\right)-\frac{\sum\limits_{n=0}^{N-1}\left(x[n]-Ar^{n+4}\right)^2}{2\sigma^2}$$

$$\frac{\partial\log p(X)}{\partial A}=\frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-Ar^{n+4})r^{n+4}=\frac{1}{\sigma^2}\sum_{n=0}^{N-1}x[n]r^{n+4}-\frac{1}{\sigma^2}\sum_{n=0}^{N-1}Ar^{2(n+4)}$$

$$\frac{\partial^2\log p(X)}{\partial A^2}=-\frac{1}{\sigma^2}\sum_{n=0}^{N-1}r^{2(n+4)}$$

$$\text{CRLB}=\frac{\sigma^2}{\sum\limits_{n=0}^{N-1}r^{2(n+4)}}$$

(c)

$$\frac{\partial\log p(X)}{\partial A}=\frac{1}{\sigma^2}\left[\sum_{n=0}^{N-1}x[n]r^{n+4}-\sum_{k=0}^{N-1}Ar^{2(k+4)}\right]$$

$$=\frac{\sum\limits_{k=0}^{N-1}r^{2(k+4)}}{\sigma^2}\left[\frac{\sum\limits_{n=0}^{N-1}x[n]r^{n+4}}{\sum\limits_{k=0}^{N-1}r^{2(k+4)}}-A\right]$$

Therefore, MVU estimator is

$$\hat{A}=\frac{\sum\limits_{n=0}^{N-1}x[n]r^{n+4}}{\sum\limits_{k=0}^{N-1}r^{2(k+4)}}$$

III. Linear models and best linear unbiased estimators

A. *Question 1*

We wish to estimate the amplitudes of exponentials in noise. The observed data are

$$x\left[n\right]=\sum_{i=1}^{p}A_ir_i^n+w[n],\ n=0,1,\cdots N-1$$

where $w[n]$ is white Gaussian noise with variance σ^2. Find MVU estimator of the amplitudes and also their covariance. Next, evaluate your results for the case when $p=2$, $r_1=1$, $r_2=-1$, and N is even.

Answer:

$$\boldsymbol{\theta}^T = \left[\begin{array}{ccc} A_1 & \cdots & A_p \end{array}\right]$$
$$\mathbf{H} = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_p \\ \vdots & \vdots & \cdots & \vdots \\ r_1^{N-1} & r_2^{N-1} & \cdots & r_p^{N-1} \end{array}\right]$$
$$\mathbf{x}^T = \left[\begin{array}{cccc} x[0] & x[1] & \cdots & x[N-1] \end{array}\right]$$
$$\hat{\theta} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x}$$
$$C_{\hat{\theta}} = \sigma^2\left(\mathbf{H}^T\mathbf{H}\right)^{-1}$$

For the second part of the question:

$$\boldsymbol{\theta}^T = \left[\begin{array}{cc} A_1 & A_2 \end{array}\right]$$
$$\mathbf{H} = \left[\begin{array}{cc} 1 & 1 \\ r_1 & r_2 \\ \vdots & \vdots \\ r_1^{N-1} & r_2^{N-1} \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{array}\right]$$
$$\mathbf{H}^T = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 \end{array}\right]$$
$$\mathbf{H}^T\mathbf{H} = \left[\begin{array}{cc} N & 0 \\ 0 & N \end{array}\right] = N\mathbf{I}$$
$$\hat{\theta} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x} = \tfrac{1}{N}\mathbf{H}^T\mathbf{x} = \tfrac{1}{N}\left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 \end{array}\right]\mathbf{x}$$
$$= \left[\begin{array}{c} \frac{1}{N}\sum\limits_{i=0}^{N-1} x[n] \\ \frac{1}{N}\sum\limits_{i=0}^{N-1} (-1)^n x[n] \end{array}\right]$$
$$C_{\hat{\theta}} = \sigma^2(N\mathbf{I})^{-1} = \tfrac{\sigma^2}{N}\mathbf{I}$$

B. Question 2

We observe two samples of a DC level in correlated Gaussian noise

$$x[0] = A + \ w[0]$$
$$x[1] = A + \ w[1]$$

where $\mathbf{w} = [w[0] \ w[1]]^{\mathrm{T}}$ is zero mean with covariance matrix.

$$\mathbf{C} = \sigma^2\left[\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array}\right]$$

The parameter ρ is the correlation coefficient between $w[0]$ and $w[1]$. Find the MVU estimator of A and its variance. Does the estimator depend on ρ?

Note:

In this case, it can be described as $\mathbf{x} = [x[0] \ x[1]]^{\mathrm{T}}$ where $x[n]$ is not a vector but this is also different from a DC level in AWGN since the noise $w[n]$ for $n = 0, 1$ are correlated with the correlation coefficient ρ.

If the signal $\mathbf{S}(\theta)$ of $\mathbf{x} = \mathbf{S}(\theta) + \ \mathbf{w}$ is in a linear form such as $\mathbf{S}(\theta) = \mathbf{H}\theta + \mathbf{b}$, we can calculate the CRLB by using the general linear model equations.

Linear Model:

$$\hat{\theta} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}(\mathbf{x} - \mathbf{b})$$
$$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}$$

Answer: The MVU estimator is (matrix calculations omitted, can be done by hand or with MATLAB)

$$\hat{A} = (\mathbf{H}^T\mathbf{C^{-1}H})^{-1}\mathbf{H}^T\mathbf{C^{-1}x} = \frac{1}{2}\left(x[0] + x[1]\right)$$

which is seen not to depend on ρ.

The variance of the estimator is

$$\text{Var}\left(\hat{A}\right) = (\mathbf{H}^T\mathbf{C^{-1}H})^{-1} = \sigma^2\frac{1+\rho}{2}$$

which strongly depends on ρ. When $\rho = -1$, the variance is zero because the noise terms cancel out. When $\rho = 1$, the variance is σ^2, which is the same variance as with one sample only (the second sample does not help at all due to the same noise value, i.e., $w[0] = w[1]$).

C. *Question 3*

Write the observation matrix $\mathbf{H}$ for the linear model for this model $x[n] = \theta_1 + \theta_2 n + \theta_3 n^2$, $n = 1, 2, 3, 4$. Here $\theta_1, \theta_2, \theta_3$ are the unknown parameters to be estimated and n is the time index. Write also the θ vector for the linear model.

Answer: The observation matrix is:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

and the θ vector is

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

D. *Question 4*

The observed samples $\{x[0], x[1], ..., x[N-1]\}$ are I.I.D according to the following PDFs:

1) Laplacian

$$p(x[n];\mu) = \frac{1}{2}e^{(-|x[n]-\mu|)}$$

2) Gaussian

$$p(x[n];\mu) = \frac{1}{\sqrt{2\pi}}e^{[-\frac{1}{2}(x[n]-\mu)^2]}$$

Find the BLUE of the mean μ in both cases. What can you say about the MVU estimator for μ?

Answer: Let us collect the observed samples into a vector X:

$$X = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

(a) From properties of Laplacian PDFs (or by calculating the mean and variance by integration), we know that mean of the each $x[n]$ is μ and the variance of each $x[n]$ is 2.

Therefore, we have the model

$$X = \mu\mathbf{1} + \mathbf{W}$$

where $E[W] = 0$ and $E[WW^T] = \text{Var}(w[n])\mathbf{I} = 2\mathbf{I}$.

We can express the BLUE estimator of μ as:

$$\hat{\mu}_{BLUE} = \frac{S^T C^{-1} X}{S^T C^{-1} S} = \frac{\frac{1}{2}\mathbf{1}^T X}{\frac{1}{2}\mathbf{1}^T \mathbf{1}} = \frac{x[0] + x[1] + \ldots + x[N-1]}{1_0 + 1_1 + \ldots + 1_{N-1}} = \frac{\sum\limits_{n=0}^{N-1} x[n]}{N}$$

Therefore, the BLUE estimator of μ for Laplacian distributed observations is:

$$\hat{\mu}_{BLUE} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] = \bar{X}$$

The BLUE estimator is not the MVU estimator since the distribution here is Laplacian, not Gaussian.

(b) For the Gaussian distribution, mean of each sample $x[n]$ is still μ but the variance of each samples is now 1.

Therefore, we have the model

$$X = \mu\mathbf{1} + \mathbf{W}$$

where $E[W] = 0$ and $E[WW^T] = \mathrm{Var}(w[n])\mathbf{I} = \mathbf{I}$.

Therefore, the BLUE estimator of μ for Gaussian distributed observations is:

$$\hat{\mu}_{BLUE} = \frac{S^T C^{-1} X}{S^T C^{-1} S} = \frac{\mathbf{1}^T X}{\mathbf{1}^T \mathbf{1}} = \frac{x[0] + x[1] + \ldots + x[N-1]}{1_0 + 1_1 + \ldots + 1_{N-1}} = \frac{\sum\limits_{n=0}^{N-1} x[n]}{N}$$

This is exactly the same estimator as for the Laplacian distribution. Actually, no matter what constant values $\mathrm{Var}(w[n])$ is, the BLUE estimator will be exactly the same. For the Gaussian case, the BLUE estimator is the MVU estimator since the $x[n]$ distribution is Gaussian

IV. Maximum likelihood estimation

A. *Question 1*

We observed IID samples $\{x[0], x[1], ..., x[N-1]\}$ with PDF

$$p(x[n];\lambda) = \{ \begin{array}{ll} \frac{\lambda}{2} e^{-(\lambda \ |x[n]|)}, & x > 0 \\ 0, & x < 0 \end{array}$$

Find the MLE of the unknown parameter λ.

Answer:

$$p(\mathbf{X};\lambda) = \left(\tfrac{\lambda}{2}\right)^N e^{-\lambda \sum\limits_{n=0}^{N-1} |x[n]|}$$

$$\log p(\mathbf{X};\lambda) = N \log\left(\tfrac{\lambda}{2}\right) - \lambda \sum_{n=0}^{N-1} |x[n]|$$

$$\frac{\partial \log p(\mathbf{X};\lambda)}{\partial\lambda} = \frac{N}{\lambda} - \sum_{n=0}^{N-1} |x[n]|$$

$$\frac{N}{\hat{\lambda}} - \sum_{n=0}^{N-1} |x[n]| = 0$$

$$\hat{\lambda} = \frac{N}{\sum\limits_{n=0}^{N-1} |x[n]|}$$

B. *Question 2*

The probability mass function for one observation x (a non-negative integer) is

$$p(x;\theta) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^x$$

We want to estimate unknown parameter θ (which is greater than 1) based on one observation x. Find the MLE.

Answer:

$$\begin{aligned}
&p(x;\theta) = \tfrac{1}{\theta}\left(1 - \tfrac{1}{\theta}\right)^x \\
&\log p(x;\theta) = -\log(\theta) + x\log\left(1 - \tfrac{1}{\theta}\right) \\
&\tfrac{\partial \log p(x;\theta)}{\partial\theta} = -\tfrac{1}{\theta} + x\tfrac{\theta}{\theta-1}\tfrac{1}{\theta^2} = -\tfrac{(\theta-1)}{\theta(\theta-1)} + x\tfrac{1}{\theta(\theta-1)} \\
&= -\tfrac{(\theta-1)}{\theta(\theta-1)} + x\tfrac{1}{\theta(\theta-1)} = -\tfrac{\theta-1-x}{\theta(\theta-1)} \\
&-\tfrac{\hat\theta-1-x}{\hat\theta\left(\hat\theta-1\right)} = 0 \\
&\hat\theta = x + 1
\end{aligned}$$

C. *Question 3*

The probability density function of the observed samples is

$$p(x[n];\theta) = \begin{cases} \theta\exp\left(-\theta x[n]\right) & x[n] > 0 \\ 0 & x[n] < 0 \end{cases}$$

where $n = 0, 1, \cdots, N-1$. The samples are independent. We want to estimate θ.

1) Find the CRLB.
2) Is there an unbiased estimator that reaches the CRLB?
3) Find the MLE.

Answer: Let us first find the CRLB:

$$\begin{aligned}
&p(\mathbf{X};\theta) = \theta^N \exp\left(-\theta\sum_{n=0}^{N-1} x[n]\right) \\
&\log p(\mathbf{X};\theta) = N\log(\theta) - \theta\sum_{n=0}^{N-1} x[n] \\
&\tfrac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} = \tfrac{N}{\theta} - \sum_{n=0}^{N-1} x[n] \\
&\tfrac{\partial^2 \log p(\mathbf{X};\theta)}{\partial\theta^2} = -\tfrac{N}{\theta^2} \\
&\text{CRLB} = \tfrac{\theta^2}{N}
\end{aligned}$$

Then lets check does there exist an efficient estimator. There is, if we can write the derivatitave in this form:

$$\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} = I(\theta)\left(\hat\theta - \theta\right)$$

$$\frac{\partial \log p(\mathbf{X};\theta)}{\partial\theta} = \frac{1}{\theta}\left(N - \theta\sum_{n=0}^{N-1} x[n]\right) = \frac{\sum_{n=0}^{N-1} x[n]}{\theta}\left(\frac{N}{\sum_{n=0}^{N-1} x[n]} - \theta\right)$$

But $I(\theta)$ cannot be function of x! Therefore, there does not exist an efficient estimator.

Next, let us find the MLE by setting the first derivate to zero and solving:

$$\frac{\partial \log p(\mathbf{X};\theta)}{\partial \theta} = \frac{N}{\theta} - \sum_{n=0}^{N-1} x[n]$$
$$\frac{N}{\hat{\theta}} - \sum_{n=0}^{N-1} x[n] = 0$$
$$\hat{\theta} = \frac{N}{\sum_{n=0}^{N-1} x[n]}$$

D. Question 4

Derive the MLE for unknown parameter θ based on independent measurements $x[1], x[2], \cdots, x[N]$, which follow the uniform distribution with range $(0, \theta)$ [the values range from 0 to θ].

Answer:

$$p(x[n];\theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x[n] \leq \theta \\ 0 & \text{otherwise} \end{cases}$$
$$p(\mathbf{X};\theta) = \begin{cases} \frac{1}{\theta^N} & 0 \leq x[n] \leq \theta, \forall n \\ 0 & \text{otherwise} \end{cases}$$

This is the maximized by minimizing $\hat{\theta}$. But there is limit how small we can make $\hat{\theta}$ since

$$\theta \geq x[n], \forall n$$
$$\theta \geq \max(\mathbf{X})$$

Therefore, the MLE is

$$\hat{\theta} = \max(\mathbf{X})$$

REFERENCES

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