Statistical Signal Processing 1, Minor exam #3, 21-Nov-2023, 10:20-11:45

Instructions

Solve all three (3) problems. No material is allowed (no calculator etc.) except for two handwritten A4 papers (totally 4 A4 pages). Please make sure to return your solutions on time. Write your name and your student ID on all papers. Clearly indicate which solution you are solving. Intermediate steps need to be included (do not just give the answer). Simplify your solutions to the extent possible. Return the exam answer paper(s) to the invigilator before leaving the exam hall. Show the student card (or mobile student card on the mobile phone) or identity card (passport, ID card, or KELA card with a photo) to the exam invigilator.

A. *Problem 1*

For the signal model

$$s[n] = \begin{cases} 3A & 0 \le n \le M-1 \\ -3A & M \le n \le N-1 \end{cases}$$

derive the least squares estimator of A. Assume received noisy samples are $x[n] = s[n] + w[n], n = 0, 1, \cdots, N-1$. Noise is zero mean. Intermediate steps need to be included.

Solution: To find LSE we need to minimize

$$J = \sum_{n=0}^{N-1} (x[n] - s[n])^2 = \sum_{n=0}^{M-1} (x[n] - 3A)^2 + \sum_{n=M}^{N-1} (x[n] + 3A)^2$$

In order to do that, let us take derivate with respect to A

$$\begin{aligned} \frac{\partial J}{\partial A} &= -6 \sum_{n=0}^{M-1} (x[n] - 3A) + 6 \sum_{n=M}^{N-1} (x[n] + 3A) \\ &= -6 \sum_{n=0}^{M-1} x[n] + 18AM + 6 \sum_{n=M}^{N-1} x[n] + 18A (N - M) \end{aligned}$$

Set the derivative to zero to get the LSE

$$\begin{aligned} -6 \sum_{n=0}^{M-1} x[n] + 18\hat{A}M + 6 \sum_{n=M}^{N-1} x[n] + 18\hat{A} (N - M) = 0 \\ \hat{A} (18M + 18 (N - M)) = 6 \sum_{n=0}^{M-1} x[n] - 6 \sum_{n=M}^{N-1} x[n] \\ \hat{A} = \frac{6 \sum_{n=0}^{M-1} x[n] - 6 \sum_{n=M}^{N-1} x[n]}{18M + 18(N-M)} = \frac{\sum_{n=0}^{M-1} x[n] - \sum_{n=M}^{N-1} x[n]}{3N} \end{aligned}$$

B. *Problem 2*

Find MAP estimator of random parameter θ when the joint PDF between measurements $z[k]$ and θ is

$$p(z[k], \theta) = \theta \exp(-\theta z[k])$$

where $\theta > 0, z[k] > 1$ and $k = 1, 2, \cdots, N$. Measurements $z[k]$ are **conditionally** independent (given θ). Hint: As an example consider variables a, b, c. We say that a and b are conditionally independent given c if

$$p(a, b|c) = p(a|c) p(b|c)$$

Solution: First, we need to find

$$p\left(z[k]\right|\theta) = \frac{p\left(z[k],\theta\right)}{p\left(\theta\right)}$$

where

$$p\left(\theta\right) = \int_{1}^{\infty} \theta \exp\left(-\theta x\right) dx = \theta \int_{1}^{\infty} \exp\left(-\theta x\right) dx = \exp\left(-\theta\right)$$

Now,

$$p\left(z[k]\right|\theta) = \frac{\theta \exp\left(-\theta z[k]\right)}{\exp\left(-\theta\right)}$$

Now, the conditional PDF of the vector of the observations $\mathbf{Z}$ is

$$p\left(\mathbf{Z}\right|\theta) = \frac{\theta^N \exp\left(-\theta \sum_{k=1}^{N} z[k]\right)}{\exp\left(-N\theta\right)}$$

Now, the MAP estimator is found with

$$\hat{\theta} = \arg\max_{\theta} p\left(\mathbf{Z}\right|\theta) p\left(\theta\right) = \arg\max_{\theta} \frac{\theta^N \exp\left(-\theta \sum_{k=1}^{N} z[k]\right)}{\exp\left(-\left(N-1\right)\theta\right)}$$

To solve this, let us the logarithm

$$\log\left(p\left(\mathbf{Z}\right|\theta) p\left(\theta\right)\right) = N \log\left(\theta\right) - \theta \sum_{k=1}^{N} z[k] + \left(N-1\right)\theta$$

and next take the derivative

$$\frac{\partial \log\left(p\left(\mathbf{Z}\right|\theta) p\left(\theta\right)\right)}{\partial \theta} = \frac{N}{\theta} - \sum_{k=1}^{N} z[k] + \left(N-1\right)$$

Setting the derivative to zero and solving we get the MAP estimate (since the second derivative is always negative) and the estimate is always within the validity range of θ:

$$\hat{\theta} = \frac{N}{\sum_{k=1}^{N} z[k] - \left(N-1\right)}$$

C. Problem 3

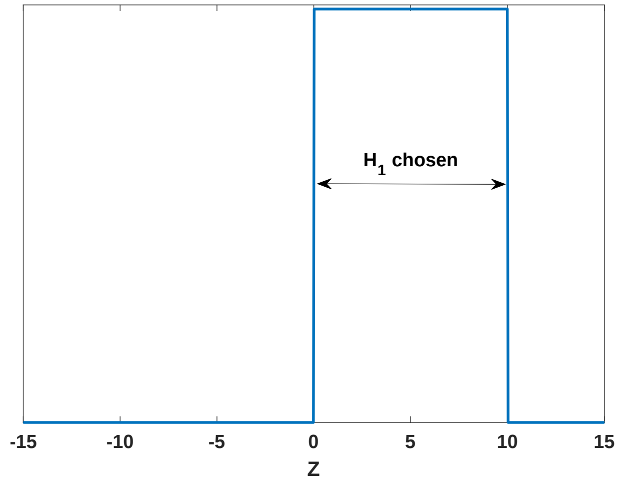
Let us assume that the random variable Z follows the Cauchy distribution

$$p\left(Z\right|\theta) = \frac{1}{\pi\left(1 + \left(Z-\theta\right)^2\right)}$$

There is only one observation Z. First, find the MAP (minimun probability of error) decision rule (in as simplified form as possible) when hypothesis are

$$\begin{aligned} H_0 &: \theta = -1 \\ H_1 &: \theta = 1 \end{aligned}$$

and $P(H_0) = 5/6$. Next, make a plot illustrating for what values of Z hypothesis H_1 is chosen (can be solved without calculator). An example plot is given below (make sure that the limits for H_1 range are clear).



Solution: We know that the MAP decision rule is

$$\frac{p\left(x|\,H_1\right)}{p\left(x|\,H_0\right)} > \frac{P\left(H_0\right)}{P\left(H_1\right)}$$

By plugging in the PDFs we get

$$\frac{\frac{1}{\pi\left(1+(Z-1)^2\right)}}{\frac{1}{\pi\left(1+(Z+1)^2\right)}} > \frac{P\left(H_0\right)}{P\left(H_1\right)}$$

which can be simplified to

$$\frac{Z^2+2Z+2}{Z^2-2Z+2} > \frac{P\left(H_0\right)}{P\left(H_1\right)}$$

which can be written as

$$-4Z^2+12Z-8>0$$

When the left hand side of the above expression gets a positive value hypothesis H_1 is chosen. We can find the roots of quadratic equation by hand (using the equation provided for the exam) and get that the roots are $Z=1$ and $Z=2$. By testing values in each segment, we find out that H_1 is chosen when $1<Z<2$.

$$f(\theta \mid x) = \frac{f(\theta, x)}{f(x)} = \frac{f(\theta)\, f(x \mid \theta)}{f(x)} \qquad \frac{P(H_0)}{P(H_1)}$$

$$\int Ap(A \mid \mathbf{x})dA \qquad L(x) = \frac{p(x; H_1)}{p(x; H_0)} > \gamma$$

$$p(A \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid A)p(A)}{p(\mathbf{x})}$$

$$= \frac{p(\mathbf{x} \mid A)p(A)}{\int p(\mathbf{x} \mid A)p(A)dA} \qquad \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x}$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta}\,[p(\mathbf{x} \mid \theta)\; p(\theta)]$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$