Statistical Signal Processing 1, Minor exam #2, 16-Nov-2023, 10:20-11:45

Instructions

Solve all three (3) problems. No material is allowed (no calculator etc.) except for two handwritten A4 papers (totally 4 A4 pages). Please make sure to return your solutions on time. Write your name and your student ID on all papers. Clearly indicate which solution you are solving. Intermediate steps need to be included (do not just give the answer). Simplify your solutions to the extent possible. Return the exam answer paper(s) to the invigilator before leaving the exam hall. Show the student card (or mobile student card on the mobile phone) or identity card (passport, ID card, or KELA card with a photo) to the exam invigilator.

A. *Problem 1*

The data

$$x[n] = -3A + w[n]$$

for $n = 0, 1, \cdots, N-1$ is observed where $w[n]$ is zero-mean white Gaussian noise with variance σ^2. We want to estimate unknown parameter A based on the observations $x[n]$.

1) Find the CRLB for A
2) Find the efficient estimator.
3) Assume that $B = -3A$ (transformation of parameters). Find CRLB for B.

Solution: By using the theory of signal in white Gaussian noise:

$$\mathrm{CRLB} = \frac{\sigma^2}{\sum_{n=0}^{N-1}\left(\frac{\partial(-3A)}{\partial A}\right)^2} = \frac{\sigma^2}{\sum_{n=0}^{N-1}(-3)^2} = \frac{\sigma^2}{9N}$$

By using the standard approach:

$$\begin{aligned}
&p(\mathbf{X}) = \tfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\tfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]+3A)^2\right)\\
&\log p(\mathbf{X}) = -\tfrac{N}{2}\log(2\pi\sigma^2) - \tfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]+3A)^2\\
&\tfrac{\partial \log p(\mathbf{X})}{\partial A} = -\tfrac{3}{\sigma^2}\sum_{n=0}^{N-1}(x[n]+3A)\\
&\tfrac{\partial^2 \log p(\mathbf{X})}{\partial A^2} = -\tfrac{9N}{\sigma^2}\\
&\mathrm{CRLB} = \tfrac{\sigma^2}{9N}
\end{aligned}$$

To get efficient estimator, we use the first derivative:

$$\begin{aligned}
\tfrac{\partial \log p(\mathbf{X})}{\partial A} &= -\tfrac{3}{\sigma^2}\sum_{n=0}^{N-1}(x[n]+3A) = -\tfrac{3}{\sigma^2}\sum_{n=0}^{N-1}x[n] - \tfrac{3}{\sigma^2}\sum_{n=0}^{N-1}3A\\
&= -\tfrac{3}{\sigma^2}\sum_{n=0}^{N-1}x[n] - \tfrac{9AN}{\sigma^2} = \tfrac{9N}{\sigma^2}\left(-\tfrac{1}{3N}\sum_{n=0}^{N-1}x[n] - A\right)
\end{aligned}$$

Thefore, we get that the efficient estimator is

$$\hat{A} = -\frac{1}{3N}\sum_{n=0}^{N-1}x[n]$$

By transformation of parameters, $B = g(A)$, where $g(A) = -3A$, we get that

$$CLRB_B = \left(\frac{\partial g(A)}{\partial A}\right)^2 CRLB_A = (-3)^2 CRLB_A = \frac{\sigma^2}{N}$$

B. Problem 2

The data $x[n] = A + B\exp(-2n) + w[n]$ for $n = 0, 1, \cdots, N-1$ is received where $w[n]$ is zero-mean white Gaussian noise with variance σ^2. We want to estimate unknown parameters A and B. Assume that $N = 4$.

1) Write the linear model for this problem including observation matrix H, vector $\boldsymbol{\theta}$, and noise vector $\mathbf{w}$ (and its covariance matrix $\mathbf{C}$).
2) How would you mathematically check (give equation) which parameter can be estimated more accurately (with less variance) for this problem?

Solution:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & \exp(-2) \\ 1 & \exp(-4) \\ 1 & \exp(-6) \end{bmatrix}$$

$$\boldsymbol{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} w[0] \\ w[1] \\ w[2] \\ w[3] \end{bmatrix}$$

$$\mathbf{C} = \sigma^2 \mathbf{I}$$

Regarding second part of the question: The covariance matrix (of size 2×2) of the estimates is:

$$\mathbf{C}_{\hat{\theta}} = \sigma^2 \left(\mathbf{H}^T \mathbf{H}\right)^{-1}$$

The variance for A is

$$Var(A) = \mathbf{C}_{\hat{\theta}}[1,1]$$

(the first diagonal element of $\mathbf{C}_{\hat{\theta}}$) and the variance for B is

$$Var(B) = \mathbf{C}_{\hat{\theta}}[2,2]$$

(the second diagonal element). The parameter with lower variance can be estimated more accurately (by the definition given).

C. Problem 3

There are independent measurements $x[n]$, $n = 1, 2, \cdots, N$ that follow the distribution:

$$p(x[i];\theta) = \theta^2 x[i] \exp(-\theta x[i])$$

where $x[i] > 0, \theta > 0$.

1) Find the MLE for θ.
2) Find the MLE for $B = \sin(\theta)$. Justify your answer (only correct answer is zero points).

Solution:

$$p(\mathbf{X};\theta) = \theta^{2N} \prod_{i=1}^{N} x[i] \exp\left(-\theta \sum_{i=1}^{N} x[i]\right)$$
$$\log p(\mathbf{X};\theta) = 2N \log(\theta) + \sum_{i=1}^{N} \log(x[i]) - \theta \sum_{i=1}^{N} x[i]$$
$$\frac{\partial \log p(\mathbf{X};\theta)}{\partial \theta} = \frac{2N}{\theta} - \sum_{i=1}^{N} x[i]$$
$$\frac{2N}{\hat{\theta}} - \sum_{i=1}^{N} x[i] = 0$$
$$\hat{\theta} = \frac{2N}{\sum_{i=1}^{N} x[i]} = \frac{2}{\frac{1}{N}\sum_{i=1}^{N} x[i]} = \frac{2}{\bar{x}}$$

By the invariance principle, $\hat{B} = \sin(\hat{\theta})$