Statistical Signal Processing 1, Minor exam #1, 05-Oct-2023, 10:20-11:45

INSTRUCTIONS

Solve all three (3) problems. No material is allowed (no calculator etc.) except for two handwritten A4 papers (totally 4 A4 pages). Please make sure to return your solutions on time. Write your name and your student ID on all papers. Clearly indicate which solution you are solving. Intermediate steps need to be included (do not just give the answer). Simplify your solutions to the extent possible. Return the exam answer paper(s) to the invigilator before leaving the exam hall. Show the student card (or mobile student card on the mobile phone) or identity card (passport, ID card, or KELA card with a photo) to the exam invigilator.

A. Problem 1

We use estimator

$$\widehat{\theta} = \frac{1}{2N}\sum_{i=1}^{N}\left(z[i]^2 - 2\right)$$

to estimate unknown parameter θ. In addition we know that $E\left[z^2\left[i\right]\right] = 2\left(\theta+1\right)$ (where $z^2\left[i\right] = z[i]\times z[i]$). Prove that $E\left[\hat{\theta}\right] = \theta$. The samples $z[i], i = 1, 2, \cdots, N$ are not necessarily independent. Justify each step given this fact.

Solution: By the linearity of expectation we get

$$E\left[\hat{\theta}\right] = \tfrac{1}{2N}\sum_{i=1}^{N} E\left(z[i]^2 - 2\right) = \tfrac{1}{2N}\sum_{i=1}^{N}\left(E\left[z[i]^2\right] - 2\right) = \tfrac{1}{2N}\sum_{i=1}^{N}\left(2\left(\theta+1\right) - 2\right)$$
$$= \tfrac{1}{2N}\sum_{i=1}^{N}\left(2\theta + 2 - 2\right) = \tfrac{1}{2N}\sum_{i=1}^{N}\left(2\theta\right) = \tfrac{2\theta N}{2N} = \theta$$

B. Problem 2

Random variables X and Y have the joint probability density function (PDF)

$$f_{XY}\left(x,y\right) = \begin{cases} \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6} & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $0 \le y \le 2$.

- Find conditional PDF of X given $Y = y$
- Find $P\left(X < \frac{1}{2} | Y = y\right)$
- Find the expected value of $2Y - 1$, i.e, $E\left[2Y - 1\right]$. Give the solution as a rational number.

Solution: First, we need the marginal distribution of Y. By some easy integrations, we get

$$f_Y\left(y\right) = \int_0^1 \left(\frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}\right)dx = \frac{1 + 3y^2 + y}{12}$$

By the definition of conditional probability

$$f_{X|Y}\left(x|y\right) = \frac{f_{XY}\left(x,y\right)}{f_Y\left(y\right)}$$

we get

$$f_{X|Y}(x|y) = \begin{cases} \frac{12\left(\frac{x^2}{4}+\frac{y^2}{4}+\frac{xy}{6}\right)}{1+3y^2+y} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

which can be simplified to

$$f_{X|Y}(x|y) = \begin{cases} \frac{3x^2+3y^2+2xy}{1+3y^2+y} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

To answer to next sub-problem, we calculate

$$P\left(X < \tfrac{1}{2} | Y = y\right) = \int_0^{1/2} \frac{3x^2+3y^2+2xy}{1+3y^2+y} dx$$
$$= \frac{1}{1+3y^2+y} \int_0^{1/2} (3x^2+3y^2+2xy)dx = \frac{\frac{1}{8}+\frac{3}{2}y^2+\frac{y}{4}}{1+3y^2+y}$$

Finally, to answer the third sub-problem, we use the LOTUS principle and calculate

$$E\left[2Y-1\right] = \frac{1}{12}\int_0^2 (2y-1)\left(1+3y^2+y\right)dy = \frac{16}{9}$$

C. Problem 3

Assume that X_1 and X_2 are random variables with joint probability density function (PDF)

$$f_{X_1X_2}(x_1, x_2) = \exp\left(-x_1 - x_2\right), 0 < x_1 < \infty, 0 < x_2 < \infty$$

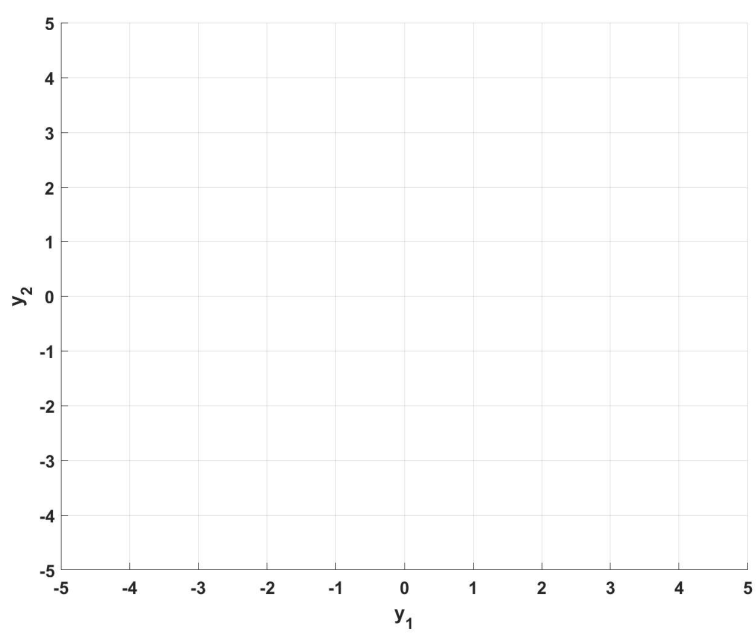
Let us consider transformation of random variables:

$$Y_1 = X_1 + X_2$$

and

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

Find the joint PDF of Y_1 and Y_2. Remember to find the validity region for the joint PDF. In addition to the equations, visually show the validity region of the joint PDF in the two dimensional xy-plane [draw a grid similar to the below figure]



Solution: We notice that

$$Y_2 = \frac{X_1}{Y_1}$$

Therefore,

$$X_1 = Y_1 Y_2$$

Therefore,

$$X_2 = Y_1 - X_1 = Y_1 - Y_1 Y_2$$

Now, we know from the theory of transformation of random variables

$$f_{Y_1 Y_2}(y_1, y_2) = \exp(-y_1 y_2 - y_1 + y_1 y_2) |J| = \exp(-y_1) |J|$$

where the Jabobian is

$$J = \det \begin{bmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{bmatrix} = \det \begin{bmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{bmatrix} = -y_1 y_2 - y_1 + y_1 y_2 = -y_1$$

Finally, we get

$$f_{Y_1 Y_2}(y_1, y_2) = \exp(-y_1 y_2 - y_1 + y_1 y_2) |J| = \exp(-y_1) y_1$$

when

$$0 < y_1 y_2 < \infty, 0 < y_1 - y_1 y_2 < \infty$$

and zero otherwise. One condition is that

$$y_1 - y_1 y_2 > 0$$

Therefore,

$$y_1 > y_1 y_2$$

But we know that $y_1 y_2 > 0$, therefore it must be that

$$y_1 > 0$$

All these values are possible. Next, we need to determine limits for y_2. From

$$y_1 > y_1 y_2$$

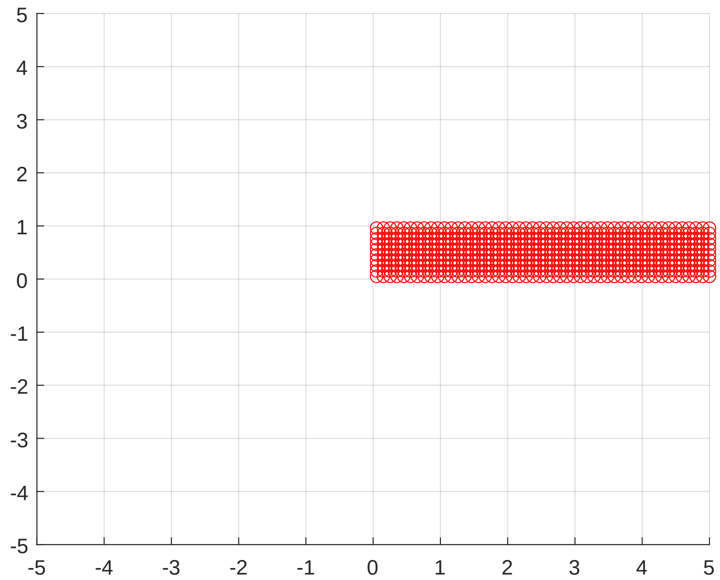
we get that

$$y_2 < 1$$

and we know that only positive values are possible for y_2. Therefore, the validity region is

$$0 < y_1 < \infty, 0 < y_2 < 1$$

which can be verified by checking that the joint PDF over this region integrates to one.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