UNIVERSITY OF OULU

CWC
oulu.fi/cwc

$\mathrm{Var}(\hat{\theta}) \geq \frac{\sigma^2}{\frac{\partial s(n;\theta)}{\partial \theta}}$

521348S STATISTICAL SIGNAL PROCESSING I

First Midterm Exam

7 – Oct – 2019

Solve all three (3) problems. Mark on the top of each paper to which problem your answer belongs to, if it continues from a different paper.

Use of function calculator is allowed, but programmable calculators not.

---

1. The observed data samples follow the model

$$x[n] = Ar^{n+4} + w[n]$$

for $n = 0, 1, 2, ...N-1$ where $r > 0$ is a known constant and additive noise $w[n]$ is zero-mean white Gaussian random process with variance σ^2. Deterministic amplitude A is an unknown parameter to be estimated.

(a) Write the log likelihood function for the observed vector x. Write all terms exactly and simplify.

(b) Derive the Cramer-Rao Lower Bound (CRLB) for the estimate of parameter A.

(c) Derive the MVU estimator for A.

Hint: You can solve (b) and (a) either using the standard approach or more easily using a linear model approach. Final answer must be simplified anyways (do not present just a matrix operation without situation). Remember to clearly write limits for the sums.

The probability density function (PDF) of the real valued normal or Gaussian distribution with mean μ and variance σ^2 is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

2. We observe two samples of an unknown DC level A in correlated Gaussian noise:

$$x[0] = A + w[0]$$

$$x[1] = A + w[1]$$

where $\mathbf{w} = [w[0]\ w[1]]^{\mathrm{T}}$ is zero mean Gaussian random vector with known covariance matrix.

$$\mathbf{C} = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Assume C is the positive definite and the parameter $-1 < \rho < 1$ is the correlation coefficient between $w[0]$ and $w[1]$. Find the MVU estimator of A and its variance (using linear model theory). Does the estimator depend on ρ?

Hint: The inverse matrix of $[AB; CD]$ with A, B, C, D scalars is

$$\begin{bmatrix} -D/(BC-AD) & B/(BC-AD) \end{bmatrix}$$
$$\begin{bmatrix} C/(BC-AD) & -A/(BC-AD) \end{bmatrix}$$

The PDF of a multivariate real valued normal or Gaussian distribution with mean μ and covariance matrix $\sigma^{\mathbf{2}}$ is

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\boldsymbol{\Sigma})} \exp[-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)]$$

3. For IID sapmles $x[n] \sim U[0, A]$, $n = 0, 1, 2, ...N-1$, A is unknown deterministic parameter to be estimated and U denotes the uniform distribution.

   (a) Find the maximum likelihood estimator of A.

   (b) Assume we use the estimator $\hat{A} = 2\bar{x}$, where $\bar{x}$ is the sample mean. Is estimator $\hat{A} = 2\bar{x}$ unbiased?

   (c) Find the distribution of the estimator $\hat{A} = 2\bar{x}$.