Signaalianalyysi 031080A

1. välikoe 27.11.2024

Välivaiheet ja perustelut näkyviin! Perustele vastauksesi myös piirrostehtävissä.

1. (a) Piirrä signaali $u(1-t)$. (1 p)

   (b) Tutki laskemalla onko signaali $x(t) = e^t u(1-t)$ energia- tai tehosignaali. (2 p)

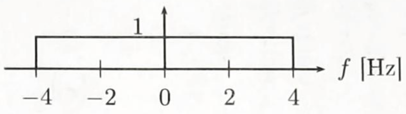
   (c) Laske signaalien $x(t) = \begin{cases} 1, & 0 < t < 1, \\ 0, & \text{muulloin,} \end{cases}$ ja $y(t) = u(t)$ konvoluutio. (3 p)

2. Näytteenotin ottaa analogisesta signaalista $x(t)$ näytteitä T sekunnin välein ja tuottaa aikadiskreetin signaalin $x[n] = x(nT)$.

   (a) Kalibroidaan laite ottamalla signaalista $\cos(8\pi t)$ näytteitä, kun $T = \frac{1}{5}$ s. Määrää näytteistetyn signaalin digitaalinen taajuus (kulmataajuus välillä $[0, \pi]$). (2 p)

   (b) Säädetään laitteen kellotaajuutta siten, että näytteenottoväliksi tulee $T = 1/6$ s, ja näytteistetään signaali $x(t)$, jonka Fourier-muunnos on kuvassa 1. Piirrä näytejonon $\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$ Fourier-muunnos. Tapahtuuko laskostumista? Mikä on signaalin $x(t)$ Nyquistin taajuus? (4 p)

Kuva 1



3. (a) Laske signaalin $x[n] = \{\underset{\uparrow}{0}, 1, 2, 1\}$ 4 pisteen diskreetti Fourier-muunnos $X[k]$ ja aikadiskreetti Fourier-muunnos $X(\omega)$. Piirrä jomman kumman amplitudispektri. Mikä yhteys muunnoksilla on?

   (b) Lineaarisen aikainvariantin järjestelmän taajuusvastefunktio on

$$H(f) = \frac{2}{3 + i2\pi f}.$$

   Mikä on järjestelmän vaste

   i. impulssiin $\delta(t)$, (1 p)

   ii. signaaliin $x(t) = \cos(2\pi f_0 t)$, kun $f_0 = \frac{3}{2\pi}$. (2 p)

4. (a) Differenssiyhtälö

$$y[n] = \frac{1}{2}y[n-1] + x[n] + x[n-1]$$

   määrittelee lineaarisen aikainvariantin systeemin, missä $x[n]$ on heräte ja $y[n]$ on vaste. Määrää systeemin siirtofunktio sekä taajuusvastefunktio jos se on olemassa. Määrää vaste herätteeseen $x[n] = (-1)^n$.

   (b) Tarkastellaan signaalia $x(t) = \text{sinc}(2t)$.

   i. Määrää signaalin $x(t)$ Fourier-muunnos.

   ii. Piirrä signaalin $x(t)$ Hilbert-muunnoksen $\hat{x}(t)$ amplitudispektri. (Merkintää $\hat{x}(t)$ ei tule sekoittaa näytejonoon, jota merkitään samalla symbolilla.)

   iii. Piirrä signaalin $x_+(t) = x(t) + i\hat{x}(t)$ amplitudispektri, missä $\hat{x}(t)$ on signaalin $x(t)$ Hilbert-muunnos.

SIGNAL ANALYSIS – FORMULA SHEET

Table A. Properties of the Fourier transform
(TD = time domain, FD = frequency domain)

Property	Mathematical description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$, where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a\rvert}G(\frac{f}{a})$, where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t-t_0) \rightleftharpoons G(f)e^{-i2\pi f t_0}$
5. Frequency shifting	$g(t)e^{i2\pi f_0 t} \rightleftharpoons G(f-f_0)$
6. Area in TD	$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$
7. Area in FD	$g(0) = \int_{-\infty}^{\infty} G(f)\,df$
8. Differentiation in TD	$\frac{d}{dt}g(t) \rightleftharpoons i2\pi f G(f)$
9. Integration in TD	$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{i2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $\overline{g(t)} \rightleftharpoons \overline{G(-f)}$
11. Multiplication in TD	$g_1(t)g_2(t) \rightleftharpoons G_1(f) * G_2(f)$
12. Convolution in TD	$g_1(t) * g_2(t) \rightleftharpoons G_1(f)G_2(f)$

Table B. Fourier transform pairs

	Time function	Fourier transform
1.	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\mathrm{sinc}(fT)$
2.	$\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
3.	$e^{-at}u(t), \quad a>0$	$\frac{1}{a+i2\pi f}$
4.	$e^{-a\lvert t\rvert}, \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
5.	$e^{-\pi t^2}$	$e^{-\pi f^2}$
6.	$\mathrm{tri}(\frac{t}{T})$	$T\mathrm{sinc}^2(fT)$
7.	$\mathrm{sinc}^2(Wt)$	$\frac{1}{W}\mathrm{tri}(\frac{f}{W})$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t-t_0)$	$e^{-i2\pi f t_0}$
11.	$e^{i2\pi f_0 t}$	$\delta(f-f_0)$
12.	$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$
13.	$\sin(2\pi f_0 t)$	$\frac{1}{2i}[\delta(f-f_0)-\delta(f+f_0)]$
14.	$\mathrm{sgn}(t)$	$\frac{1}{i\pi f}$
15.	$\frac{1}{\pi t}$	$-i\,\mathrm{sgn}(f)$
16.	$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{i2\pi f}$
17.	$\sum_{k=-\infty}^{\infty} \delta(t-kT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T_0})$

Table C. Properties of the Fourier transform for discrete time signals

Property	Time domain	Frequency domain
Notation	$x[n]$	$X(\omega)$
	$y[n]$	$Y(\omega)$
1. Linearity	$ax[n] + by[n]$	$aX(\omega) + bY(\omega)$
2. Time shifting	$x[n-k]$	$e^{-i\omega k}X(\omega)$
3. Time reversal	$x[-n]$	$X(-\omega)$
4. Differentiation in FD	$nx[n]$	$i\frac{dX(\omega)}{d\omega}$
5. Convolution	$x[n] * y[n]$	$X(\omega)Y(\omega)$
6. Correlation	$r_{xy}[l] = \overline{x[-l]} * y[l]$	$R_{xy}(\omega) = \overline{X(\omega)}Y(\omega)$
	$r_{xy}(\tau) = \overline{x(-\tau)} * y(\tau)$	$R_{xy}(f) = \overline{X(f)}Y(f)$

Table D. Trigonometric identities

1. $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$	6. $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
2. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$	7. $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
3. $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$	8. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
4. $\sin^2\theta + \cos^2\theta = 1$	9. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
5. $\cos^2\theta - \sin^2\theta = \cos(2\theta)$	10. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$

Table E. Properties of the Z-transform

Property	Time domain	z-domain (=ZD)	Region of convergence
Notation	$x[n]$	$X(z)$	$\mathrm{ROC}_x = \{z \mid r_2 < \lvert z\rvert < r_1\}$
	$y[n]$	$Y(z)$	ROC_y
1. Linearity	$ax[n] + by[n]$	$aX(z) + bY(z)$	At least the intersection $\mathrm{ROC}_x \cap \mathrm{ROC}_y$
2. Time shifting	$x[n-k]$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$, if $k > 0$ and $z = \infty$, if $k < 0$
3. Scaling in ZD	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
4. Time reversal	$x[-n]$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
5. Differentiation in ZD	$nx[n]$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
6. Convolution	$x[n] * y[n]$	$X(z)Y(z)$	At least $\mathrm{ROC}_x \cap \mathrm{ROC}_y$

Table F. Z-transform pairs

Signal $x[n]$	z-transform $X(z)$	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
4. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

Table G. Functions

1. Rectangular function	$\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \lvert t\rvert > \frac{1}{2} \end{cases}$
2. Triangular function	$\mathrm{tri}(t) = \begin{cases} 1-\lvert t\rvert, & \lvert t\rvert < 1 \\ 0, & \lvert t\rvert > 1 \end{cases}$
3. Unit step function (cont.)	$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$
4. Unit step function (discr.)	$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
5. Signum function	$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$
6. Dirac delta function	$\delta(t) = 0,\ t \neq 0, \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$
or equivalently	$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$
7. Discrete delta function	$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$
8. Sinc function	$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

Table H. Definitions

1. $E_x = \int\limits_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$
2. $P_x = \lim\limits_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \lvert x(t)\rvert^2 dt$
3. $E_x = \sum\limits_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$
4. $P_x = \lim\limits_{M\to\infty} \frac{1}{2M+1} \sum\limits_{n=-M}^{M} \lvert x[n]\rvert^2$
5. $r_{xy}[m] = \sum\limits_{n=-\infty}^{\infty} \overline{x[n]}y[n+m]$
6. $r_{xy}(\tau) = \int\limits_{-\infty}^{\infty} \overline{x(t)}y(t+\tau)\,dt$
7. $x[n] * y[n] = \sum\limits_{k=-\infty}^{\infty} x[k]y[n-k]$
8. $x(t) * y(t) = \int\limits_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau$
9. $X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\,dt$
10. $x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}\,df$
11. $X[k] = \sum\limits_{n=0}^{N-1} x[n]e^{-i2\pi kn/N}$
12. $x[n] = \frac{1}{N}\sum\limits_{k=0}^{N-1} X[k]e^{i2\pi kn/N}$
13. $X(\omega) = \sum\limits_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$
14. $x[n] = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} X(\omega)e^{i\omega n}\,d\omega$
15. $X(z) = \sum\limits_{n=-\infty}^{\infty} x[n]z^{-n}$
16. $x[n] = \frac{1}{2\pi i}\int\limits_{S_r} X(z)z^{n-1}\,dz$
17. $Ae^{i\omega_0 n} \to AH(\omega_0)e^{i\omega_0 n}$

Table I. Fourier transform pairs

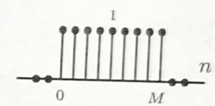
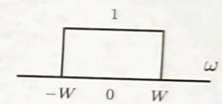
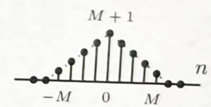
$g[n]$	$G(\omega), \quad -\pi < \omega < \pi$
1. 	$\frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}e^{-i\omega M/2}$
2. $\begin{cases} \frac{\sin(Wn)}{\pi n}, & n \neq 0 \\ \frac{W}{\pi}, & n = 0 \end{cases}, \quad \lvert W\rvert < \pi$	
3. 	$\left(\frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}\right)^2$
4. $a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1-ae^{-i\omega}}$
5. $a^{\lvert n\rvert}, \quad \lvert a\rvert < 1$	$\frac{1-a^2}{1-2a\cos(\omega)+a^2}$
6. $\delta[n]$	1
7. 1	$2\pi\delta(\omega)$
8. $\delta[n-M]$	$e^{-i\omega M}$
9. $e^{i\omega_0 n}, \quad \lvert\omega_0\rvert < \pi$	$2\pi\delta(\omega-\omega_0)$
10. $\cos(\omega_0 n), \quad \lvert\omega_0\rvert < \pi$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$
11. $\sin(\omega_0 n), \quad \lvert\omega_0\rvert < \pi$	$\frac{\pi}{i}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$
12. $u[n]$	$\pi\delta(\omega) + \frac{1}{1-e^{-i\omega}}$

Table J. Definitions & formulae

1. $\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\,d\tau, \quad \hat{X}(f) = -i\,\mathrm{sgn}(f)X(f)$
2. $x_+(t) = x(t) + i\hat{x}(t) = \tilde{x}(t)e^{i2\pi f_c t}$
3. $X \sim \mathrm{Poi}(\lambda),\ \mu = \lambda,\ \sigma^2 = \lambda,\ P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda},\ k = 0, 1, 2, \ldots$
4. $X \sim \mathrm{U}\,(a,b),\ \mu = \frac{a+b}{2},\ \sigma^2 = \frac{(b-a)^2}{12},\ f(x) = \frac{1}{b-a},\ a < x < b$
5. $X \sim \mathrm{Exp}(a),\ \mu = \frac{1}{a},\ \sigma^2 = \frac{1}{a^2},\ f(x) = ae^{-ax},\ x \geq 0$
6. $X \sim \mathrm{N}(\mu, \sigma^2),\ f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
7. $X \sim \mathrm{Rayleigh}(\alpha),\ \mu = \sqrt{\frac{\pi}{2\alpha}},\ \sigma^2 = \frac{4-\pi}{2\alpha},\ f(x) = \alpha x e^{-\frac{1}{2}\alpha x^2},\ x \geq 0$
8. $\mathrm{Var}\,(X) = E(X^2) - \mu_X^2,\ \mathrm{Cov}\,(X,Y) = E(XY) - \mu_X\mu_Y$
9. $h(u,v) = f(k_1(u,v), k_2(u,v)) \cdot \frac{1}{|J|},\ J = \frac{\partial(u,v)}{\partial(x,y)}$
10. $\{E[XY]\}^2 \leq E[X^2]E[Y^2]$
11. $C_Y = AC_XA^T$
12. $W(t) - W(s) \sim \mathrm{N}(0, \sigma^2(t-s))$
13. $X(t) - X(s) \sim \mathrm{Poi}[\lambda(t-s)]$
14. $X(t) = Y(t-A),\ A \sim \mathrm{U}\,(0,\Delta),\ R_X(\tau) = 1 - |\tau|/\Delta,\ |\tau| \leq \Delta$
15. $X(t) = N(0,t),\ P(N(0,t) = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$
16. $Z(t) = AY(t),\ R_Z(\tau) = e^{-2\lambda|\tau|}$
17. $E[(\int_a^b X(t)dt)^2] = \int_a^b\int_a^b R_X(t_1,t_2)dt_1dt_2$
18. $S_X(f) = \lim_{T\to\infty} \frac{E[|\mathcal{X}_T(f)|^2]}{2T}, \quad \langle R_X(t,t+\tau)\rangle = \int_{-\infty}^{\infty} S_X(f)e^{i2\pi f\tau}df$
19. $S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f\tau}d\tau, \quad R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{i2\pi f\tau}df$
20. $S_X(f) = \sum_{k=-\infty}^{\infty} R_X[k]e^{-i2\pi fk}, \quad R_X[k] = \int_{-1/2}^{1/2} S_X(f)e^{i2\pi fk}df$
21. $S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X[k]e^{-i\omega k}, \quad R_X[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\omega)e^{i\omega k}d\omega$
22. $P_X = \int_{-\infty}^{\infty} S_X(f)df$
23. $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(s)X(t-s)ds = \int_{-\infty}^{\infty} h(t-s)X(s)ds$
24. $Y[n] = h[n] * X[n] = \sum_{j=-\infty}^{\infty} h[j]X[n-j]$
25. $H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt$
26. $H(f) = \sum_{n=-\infty}^{\infty} h[n]e^{-i2\pi fn}, \quad H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-i\omega n}$
27. $R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$
28. $S_Y(f) = |H(f)|^2S_X(f), \quad S_Y(\omega) = |H(\omega)|^2S_X(\omega)$
29. $R_{XY}(\tau) = R_X(\tau) * h(\tau)$
30. $S_{XY}(f) = H(f)S_X(f), \quad S_{XY}(\omega) = H(\omega)S_X(\omega)$
31. $S_{YX}(f) = \overline{S_{XY}(f)} = \overline{H(f)}S_X(f), \quad S_{YX}(\omega) = \overline{S_{XY}(\omega)} = \overline{H(\omega)}S_X(\omega)$
32. $H_{\mathrm{opt}}(f) = \frac{1}{C}\frac{\overline{X(f)}}{S_N(f)}e^{-i2\pi ft_0}$
33. $H_{\mathrm{opt}}(f) = \frac{S_{XZ}(f)}{S_X(f)},\ X(t) = Z(t) + N(t)$
34. $H_{\mathrm{opt}}(f) = \frac{S_Z(f)}{S_Z(f)+S_N(f)},\ \epsilon = \int_{-\infty}^{\infty} \frac{S_Z(f)S_N(f)}{S_Z(f)+S_N(f)}df$
35. $Y[n] = \sum_{k=a}^{b} h[k]X[n-k],\ Y(t) = \int_a^b h(s)X(t-s)ds$
36. $X[k] = Z[k] + N[k],\ X(t) = Z(t) + N(t)$
37. $R_{XZ}[m] = \sum_{k=a}^{b} h[k]R_X[m-k],\ \epsilon = R_Z(0) - \sum_{k=a}^{b} h[k]R_{XZ}[k]$
38. $R_{XZ}(\tau) = \int_a^b h(s)R_X(\tau-s)ds,\ \epsilon = R_Z(0) - \int_a^b h(s)R_{XZ}(s)ds$
39. $R_{XZ}[m] = \sum_{k=0}^{\infty} h[k]R_X[m-k]$
40. $R_{XZ}(\tau) = \int_0^{\infty} h(s)R_X(\tau-s)ds$
41. $W(f) = 1/G(f),\ S_X(f) = G(f)\overline{G(f)}$