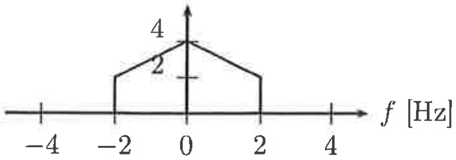
Signaalianalyysi 031080A

Loppukoe 17.1.2019

1. (a) Tutki laskemalla, onko signaali $x(t) = e^{(-1+i)t}u(t)$ energia- tai tehosignaali. Laske signaalin $y[n] = \{\underset{\uparrow}{1}, 2, 3\}$ autokorrelaatiofunktio.

   (b) Analogisen signaalin $x(t)$ Fourier-muunnos on esitetty oheisessa kuvassa.



   Kuinka pieni pitää näytteenottovälin T olla, jotta $x(t)$ voidaan yksikäsitteisesti määrätä näytteistä $x(nT)$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$? Olkoon näytteenottotaajuus $f_s = 5$ Hz. Piirrä impulssijonon $\sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$ Fourier-muunnos. Tapahtuuko laskostumista?

2. (a) Laske aikadiskreetin signaalin $x[n] = \{\underset{\uparrow}{2}, 0, 3, 1\}$ neljän pisteen diskreetti Fourier-muunnos $X[k]$, $k = 0, 1, 2, 3$. Signaali $x[n]$ on saatu näytteistämällä analoginen signaali $x(t)$ Nyquistin taajuudella ajanhetkillä $t = nT$, missä $T = 0.015625$ s. Mitä näytteistetyn signaalin analogisia taajuuksia $X[k]$:n arvot vastaavat?

   (b) Analoginen LTI-systeemi on määritelty differentiaaliyhtälöllä

$$y'(t) + 7y(t) = 2x(t-3), t \geq 0$$

   alkuehdolla $y(0) = 0$, $x(t) = 0$, $t \leq 0$, missä $x(t)$ on heräte ja $y(t)$ on vaste. Määrää tämän systeemin siirtofunktio ja impulssivaste. Onko systeemi kausaalinen?

3. Olkoon A ja Θ riippumattomia satunnaismuuttujia. A:n jakauman (piste)todennäköisyysfunktio on annettu oheisessa taulukossa ja Θ noudattaa tasajakaumaa Tas$(0, 2\pi)$

a_k	1	3	5	7
$P(A = a_k)$	0.3	0.4	0.2	0.1

   (a) Laske A:n odotusarvo $E(A)$ ja varianssi $D^2(A)$.

   (b) Laske signaalin $X(t) = A\cos(t + \Theta)$ odotusarvofunktio $\mu_X(t)$ ja autokorrelaatiofunktio $R_X(t, t+\tau)$. Mikä on signaalin $X(t)$ keskimääräinen teho?

4. (a) Aikadiskreetti LTI-systeemi määritellään yhtälöllä

$$y[n] = x[n] + x[n-1],$$

   missä $x[n]$ on heräte ja $y[n]$ vaste. Määrää siirtofunktio $H(z)$ ja taajuusvastefunktio $H(\omega)$. Olkoon sitten heräte diskreettiä valkoista kohinaa $W[n]$, jonka autokorrelaatiofunktio on

$$R_W[k] = \begin{cases} 2, & k = 0 \\ 0 & \text{muulloin.} \end{cases}$$

   Määrää vasteen $Y[n] = W[n] + W[n-1]$ autokorrelaatiofunktio ja tehotiheysspektri.

   (b) Määrää sellainen kausaalinen analoginen LTI-systeemi, jonka vaste on valkoista kohinaa tehotiheydellä 1, kun herätteen $X(t)$ tehotiheysspektri on

$$S_X(f) = 1 - \frac{5}{9 + 4\pi^2 f^2}.$$

   Anna systeemin siirtofunktio $H(f)$ ja impulssivaste $h(t)$.

ble E. Properties of the Z-transform

Property	Time domain	z-domain (=ZD)	Region of convergence
Notation	$x[n]$	$X(z)$	$\mathrm{ROC}_x = \{z \mid r_2 < \lvert z\rvert < r_1\}$
	$y[n]$	$Y(z)$	ROC_y
1. Linearity	$ax[n] + by[n]$	$aX(z) + bY(z)$	At least the intersection $\mathrm{ROC}_x \cap \mathrm{ROC}_y$
2. Time shifting	$x[n-k]$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$, if $k > 0$ and $z = \infty$, if $k < 0$
3. Scaling in ZD	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
4. Time reversal	$x[-n]$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
5. Differentiation in ZD	$nx[n]$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
6. Convolution	$x[n] * y[n]$	$X(z)Y(z)$	At least $\mathrm{ROC}_x \cap \mathrm{ROC}_y$

Table F. Z-transform pairs

Signal $x[n]$	z-transform X(z)	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
4. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

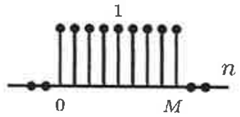
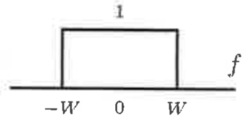
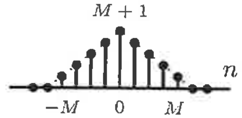
Table G. Functions

1. Rectangular function	$\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \lvert t\rvert > \frac{1}{2} \end{cases}$
2. Triangular function	$\mathrm{tri}(t) = \begin{cases} 1-\lvert t\rvert, & \lvert t\rvert < 1 \\ 0, & \lvert t\rvert > 1 \end{cases}$
3. Unit step function (cont.)	$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$
4. Unit step function (discr.)	$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
5. Signum function	$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$
6. Dirac delta function	$\delta(t) = 0,\ t \neq 0,\quad \int_{-\infty}^{\infty} \delta(t)dt = 1$
or equivalently	$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$
7. Discrete delta function	$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$
8. Sinc function	$\mathrm{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$

Table H. Definitions

1. $E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$
2. $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$
3. $E_x = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$
4. $P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \lvert x[n]\rvert^2$
5. $r_{xy}[m] = \sum_{n=-\infty}^{\infty} \overline{x[n]}y[n+m]$
6. $x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$
7. $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau$
8. $X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\,dt$
9. $x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}\,df$
10. $X[k] = \sum_{n=0}^{N-1} x[n]e^{-i2\pi kn/N}$
11. $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{i2\pi kn/N}$
12. $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$
13. $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{i\omega n}\,d\omega$
14. $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
15. $x[n] = \frac{1}{2\pi i} \int_{S_r} X(z)z^{n-1}\,dz$
16. $Ae^{i\omega_0 n} \to AH(\omega_0)e^{i\omega_0 n}$

Table I. Fourier transform pairs

$g[n]$	$G(f),\quad -\frac{1}{2} < f < \frac{1}{2}$
1. 	$\dfrac{\sin(\pi f(M+1))}{\sin(\pi f)} e^{-i\pi fM}$
2. $\dfrac{\sin(2\pi Wn)}{\pi n}$, $\lvert W\rvert < \dfrac{1}{2}$	
3. 	$\left(\dfrac{\sin(\pi f(M+1))}{\sin(\pi f)}\right)^2$
4. $a^n u[n]$, $\lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-i2\pi f}}$
5. $a^{\lvert n\rvert}$, $\lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos(2\pi f)+a^2}$
6. $\delta[n]$	1
7. 1	$\delta(f)$
8. $\delta[n-M]$	$e^{-i2\pi fM}$
9. $e^{-i2\pi f_0 n}$, $\lvert f_0\rvert < \frac{1}{2}$	$\delta(f-f_0)$
10. $\cos(2\pi f_0 n)$, $\lvert f_0\rvert < \frac{1}{2}$	$\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$
11. $\sin(2\pi f_0 n)$, $\lvert f_0\rvert < \frac{1}{2}$	$\frac{1}{2i}(\delta(f-f_0) - \delta(f+f_0))$
12. $u[n]$	$\frac{1}{2}\delta(f) + \dfrac{1}{1-e^{-i2\pi f}}$

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6. Convolution	$x[n] * y[n]$	$X(z)Y(z)$	At least $\text{ROC}_x \cap \text{ROC}_y$

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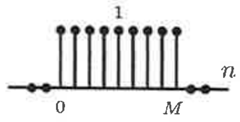
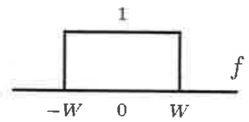
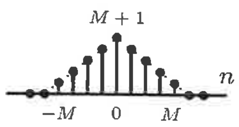
1. 	$\frac{\sin(\pi f(M+1))}{\sin(\pi f)}e^{-i\pi fM}$
2. $\frac{\sin(2\pi Wn)}{\pi n}, \quad \lvert W\rvert < \frac{1}{2}$	
3. 	$\left(\frac{\sin(\pi f(M+1))}{\sin(\pi f)}\right)^2$
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5. $a^{\lvert n\rvert}, \quad \lvert a\rvert < 1$	$\frac{1-a^2}{1-2a\cos(2\pi f)+a^2}$
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10. $\cos(2\pi f_0 n), \quad \lvert f_0\rvert < \frac{1}{2}$	$\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$
11. $\sin(2\pi f_0 n), \quad \lvert f_0\rvert < \frac{1}{2}$	$\frac{1}{2i}(\delta(f-f_0) - \delta(f+f_0))$
12. $u[n]$	$\frac{1}{2}\delta(f) + \frac{1}{1-e^{-i2\pi f}}$

Table J. Definitions & formulae

1. $\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\,d\tau$
2. $x_+(t) = x(t) + i\hat{x}(t) = \tilde{x}(t)e^{i2\pi f_c t}$
3. $X \sim \text{Poi}(\lambda),\ \mu = \lambda,\ \sigma^2 = \lambda,\ P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda},\ k = 0,1,2,\ldots$
4. $X \sim \text{Exp}(a),\ \mu = \frac{1}{a},\ \sigma^2 = \frac{1}{a^2},\ f(x) = ae^{-ax},\ x \geq 0$
5. $X \sim \text{N}(\mu,\sigma^2),\ f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
6. $X \sim \text{Rayleigh}(\alpha),\ \mu = \sqrt{\frac{\pi}{2\alpha}},\ \sigma^2 = \frac{4-\pi}{2\alpha},\ f(x) = \alpha x e^{-\frac{1}{2}\alpha x^2},\ x \geq 0$
7. $Z_k = \alpha Z_{k-1} \mod M$
8. $\{E[XY]\}^2 \leq E[X^2]E[Y^2]$
9. $C_Y = AC_XA^T$
10. $W(t) - W(s) \sim N(0, \sigma^2(t-s))$
11. $X(t) - X(s) \sim \text{Poi}[\lambda(t-s)]$
12. $X(t) = Y(t-A),\ A \sim \text{Tas}\ (0,\Delta),\ R_X(\tau) = 1 - |\tau|/\Delta,\ |\tau| \leq \Delta$
13. $X(t) = N(0,t),\ P(N(0,t) = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$
14. $Z(t) = AY(t),\ R_Z(\tau) = e^{-2\lambda|\tau|}$
15. $E[(\int_a^b X(t)dt)^2] = \int_a^b\int_a^b R_X(t_1,t_2)dt_1dt_2$
16. $S_X(f) = \lim_{T\to\infty} \frac{E[|\mathcal{X}_T(f)|^2]}{2T}, \qquad \langle R_X(t,t+\tau)\rangle = \int_{-\infty}^{\infty} S_X(f)e^{i2\pi f\tau}df$
17. $S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f\tau}d\tau, \qquad R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{i2\pi f\tau}df$
18. $S_X(f) = \sum_{k=-\infty}^{\infty} R_X[k]e^{-i2\pi fk}, \qquad R_X[k] = \int_{-1/2}^{1/2} S_X(f)e^{j2\pi fk}df$
19. $P_X = \int_{-\infty}^{\infty} S_X(f)df$
20. $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(s)X(t-s)ds = \int_{-\infty}^{\infty} h(t-s)X(s)ds$
21. $Y[n] = h[n] * X[n] = \sum_{j=-\infty}^{\infty} h[j]X[n-j]$
22. $H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt$
23. $H(f) = \sum_{n=-\infty}^{\infty} h[n]e^{-i2\pi fn}$
24. $R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$
25. $S_Y(f) = |H(f)|^2 S_X(f)$
26. $R_{XY}(\tau) = R_X(\tau) * h(\tau)$
27. $S_{XY}(f) = H(f)S_X(f)$
28. $S_{YX}(f) = \overline{S_{XY}(f)} = \overline{H(f)}S_X(f)$
29. $H_{\text{opt}}(f) = \frac{1}{C}\frac{\overline{X(f)}}{S_N(f)}e^{-i2\pi ft_0}$
30. $H_{\text{opt}}(f) = \frac{S_{XZ}(f)}{S_X(f)},\ X(t) = Z(t) + N(t)$
31. $H_{\text{opt}}(f) = \frac{S_Z(f)}{S_Z(f)+S_N(f)},\ \epsilon = \int_{-\infty}^{\infty} \frac{S_Z(f)S_N(f)}{S_Z(f)+S_N(f)}df$
32. $Y[n] = \sum_{k=a}^{b} h[k]X[n-k],\ Y(t) = \int_a^b h(s)X(t-s)ds$
33. $X[k] = Z[k] + N[k],\ X(t) = Z(t) + N(t)$
34. $R_{XZ}[m] = \sum_{k=a}^{b} h[k]R_X[m-k],\ \epsilon = R_Z(0) - \sum_{k=a}^{b} h[k]R_{XZ}[k]$
35. $R_{XZ}(\tau) = \int_a^b h(s)R_X(\tau-s)ds,\ \epsilon = R_Z(0) - \int_a^b h(s)R_{XZ}(s)ds$
36. $R_{XZ}[m] = \sum_{k=0}^{\infty} h[k]R_X[m-k]$
37. $R_{XZ}(\tau) = \int_0^{\infty} h(s)R_X(\tau-s)ds$
38. $W(f) = 1/G(f),\ S_X(f) = G(f)\overline{G(f)}$
39. $H_2(f) = \sum_{m=0}^{\infty} R_{X'Z}[m]e^{-i2\pi fm}$
40. $R_{X'Z}[k] = \sum_{i=0}^{\infty} w[i]R_{XZ}[k+i],\ S_{X'Z}(f) = S_{XZ}(f)/\overline{G(f)}$
41. $H(f) = W(f)H_2(f)$