TEKNIIKAN MATEMATIIKKA

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1. Määritellään diskreetti signaali $x(n) = \{\underset{\uparrow}{-3}, 0, -1, 6\}$. Laske signaalin $x(n)$

   (a) energia (1 p)

   (b) keskimääräinen teho (1 p)

   (c) autokorrelaatio $r_{xx}(n)$ (1 p)

   (d) diskreetti Fourier muunnos $X(k)$ ja amplitudispektri. (3 p)

2. (a) Analogisesta signaalista $x(t) = \sin(30000\pi t)$ otetaan näytteitä $4 \cdot 10^{-5}$ sekunnin välein. Tapahtuuko laskostumista? Mikä analoginen taajuus vastaa saatua diskreettiä signaalia? Mikä on riittävä näytteenottotaajuus laskostumisen estämiseksi?

   (b) Analogisen LTI-systeemi on määritelty differentiaaliyhtälöllä

   $$y'(t) + 7y(t) = 2x(t-5), \quad y(0) = 0 \text{ ja } x(t) = 0, t < 0.$$

   missä $x(t)$ on heräte ja $y(t)$ vaste. Määrää

   i. siirtofunktio,

   ii. amplitudivaste,

   iii. impulssivaste.

3. Olkoon $Y(t) = X(t)\cos(t + \Theta)$, missä $\Theta \sim \text{Tas}(0, 2\pi)$. $X(t)$ on Θ:sta riippumaton satunnaissignaali, jonka odotusarvofunktio on $E[X(t)] = 0$ ja autokorrelaatiofunktio on $R_X(\tau) = e^{-|\tau|}$. Laske $Y(t)$:n odotusarvofunktio, autokorrelaatiofunktio ja määrää $Y(t)$:n keskimääräinen teho. Tutki onko $Y(t)$ stationaarinen.

4. (a) Selitä mitä tarkoittaa valkoinen kohina. (1 p)

   (b) Laske tehtävän 2.(b) systeemin vasteen tehotiheysspektri ja autokorrelaatiofunktio, kun heräte on valkoista kohinaa, jonka tehotiheys on 1. (2 p)

   (c) Signaalia X_n estimoidaan aiempien näytteiden lineaarikombinaationa

   $$X_n \approx Y_n = h_1 X_{n-1} + h_2 X_{n-2}$$

   Määrää virheen $E[(X_n - Y_n)^2]$ minimoivat kertoimet h_1 ja h_2, kun signaalin X_n autokorrelaatiofunktiosta $R_X(m)$ tunnetaan $R_X(0) = 6$, $R_X(1) = 3$, $R_X(2) = 2$. (3 p)

able A. Properties of the Fourier transform
(TD = time domain, FD = frequency domain)

Property	Mathematical description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$, where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a\rvert}G(\frac{f}{a})$, where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t-t_0) \rightleftharpoons G(f)e^{-i2\pi f t_0}$
5. Frequency shifting	$g(t)e^{i2\pi f_0 t} \rightleftharpoons G(f-f_0)$
6. Area in TD	$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$
7. Area in FD	$g(0) = \int_{-\infty}^{\infty} G(f)\,df$
8. Differentiation in TD	$\frac{d}{dt}g(t) \rightleftharpoons i2\pi f G(f)$
9. Integration in TD	$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{i2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $\overline{g(t)} \rightleftharpoons \overline{G(-f)}$
11. Multiplication in TD	$g_1(t)g_2(t) \rightleftharpoons G_1(f) * G_2(f)$
12. Convolution in TD	$g_1(t) * g_2(t) \rightleftharpoons G_1(f)G_2(f)$

Table B. Fourier transform pairs

	Time function	Fourier transform
1.	$\text{rect}\left(\frac{t}{T}\right)$	$T\text{sinc}(fT)$
2.	$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
3.	$e^{-at}u(t), \quad a>0$	$\frac{1}{a+i2\pi f}$
4.	$e^{-a\lvert t\rvert}, \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
5.	$e^{-\pi t^2}$	$e^{-\pi f^2}$
6.	$\text{tri}(\frac{t}{T})$	$T\text{sinc}^2(fT)$
7.	$\text{sinc}^2(Wt)$	$\frac{1}{W}\text{tri}(\frac{f}{W})$
8.	$\delta(t)$	1
9.	1	$\delta(f)$
10.	$\delta(t-t_0)$	$e^{-i2\pi f t_0}$
11.	$e^{i2\pi f_0 t}$	$\delta(f-f_0)$
12.	$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$
13.	$\sin(2\pi f_0 t)$	$\frac{1}{2i}[\delta(f-f_0)-\delta(f+f_0)]$
14.	$\text{sgn}(t)$	$\frac{1}{i\pi f}$
15.	$\frac{1}{\pi t}$	$-i\,\text{sgn}(f)$
16.	$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{i2\pi f}$
17.	$\sum_{k=-\infty}^{\infty}\delta(t-kT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_0})$

Table C. Properties of the Fourier transform for discrete time signals

Property	Time domain	Frequency domain
Notation	$x(n)$	$X(\omega)$
	$y(n)$	$Y(\omega)$
1. Linearity	$ax(n)+by(n)$	$aX(\omega)+bY(\omega)$
2. Time shifting	$x(n-k)$	$e^{-i\omega k}X(\omega)$
3. Time reversal	$x(-n)$	$X(-\omega)$
4. Differentiation in FD	$nx(n)$	$i\frac{dX(\omega)}{d\omega}$
5. Convolution	$x(n)*y(n)$	$X(\omega)Y(\omega)$
6. Correlation	$r_{xy}(l) = x(-l)*y(l)$	$R_{xy}(\omega) = X(-\omega)Y(\omega)$

Table D. Trigonometric identities

1. $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$	6. $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
2. $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$	7. $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
3. $\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$	8. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$
4. $\sin^2\theta + \cos^2\theta = 1$	9. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$
5. $\cos^2\theta - \sin^2\theta = \cos(2\theta)$	10. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$

Table E. Properties of the z-transform

Property	Time domain	z-domain (=ZD)	Region of convergence
Notation	$x(n)$	$X(z)$	$\mathrm{ROC}_x = \{z \mid r_2 < \lvert z\rvert < r_1\}$
	$y(n)$	$Y(z)$	ROC_y
1. Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	At least the intersection $\mathrm{ROC}_x \cap \mathrm{ROC}_y$
2. Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$, if $k > 0$ and $z = \infty$, if $k < 0$
3. Scaling in ZD	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
4. Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
5. Differentiation in ZD	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
6. Convolution	$x(n) * y(n)$	$X(z)Y(z)$	At least $\mathrm{ROC}_x \cap \mathrm{ROC}_y$
7. Correlation	$r_{xy}(l) = x(-l) * y(l)$	$R_{xy}(z) = X(z^{-1})Y(z)$	At least $\mathrm{ROC}\{X(z^{-1})\} \cap \mathrm{ROC}_y$

Table F. z-transform pairs

Signal $x(n)$	z-transform X(z)	ROC
1. $\delta(n)$	1	All z
2. $u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $nu(n)$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
4. $a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

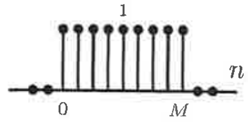
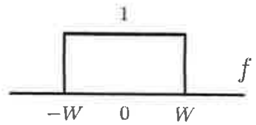
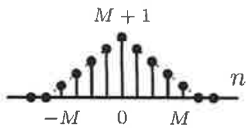
Table G. Functions

1. Rectangular function	$\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \lvert t\rvert > \frac{1}{2} \end{cases}$
2. Triangular function	$\mathrm{tri}(t) = \begin{cases} 1-\lvert t\rvert, & \lvert t\rvert < 1 \\ 0, & \lvert t\rvert > 1 \end{cases}$
3. Unit step function (cont.)	$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$
4. Unit step function (discr.)	$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
5. Signum function	$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$
6. Dirac delta function	$\delta(t) = 0,\ t \neq 0, \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$
or equivalently	$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$
7. Discrete delta function	$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$
8. Sinc function	$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

Table H. Definitions

1. $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$
2. $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$
3. $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$
4. $P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} |x(n)|^2$
5. $r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)y(n+m)$
6. $x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$
7. $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau$
8. $X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\,dt$
9. $x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}\,df$
10. $X(k) = \sum_{n=0}^{N-1} x(n)e^{-i2\pi kn/N}$
11. $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{i2\pi kn/N}$
12. $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$
13. $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}\,d\omega$
14. $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$
15. $x(n) = \frac{1}{2\pi i}\int_{S_r} X(z)z^{n-1}\,dz$
16. $Ae^{i\omega_0 n} \to AH(\omega_0)e^{i\omega_0 n}$

Table I. Fourier transform pairs

$g(n)$	$G(f), \quad -\frac{1}{2} < f < \frac{1}{2}$
1. 	$\frac{\sin(\pi f(M+1))}{\sin(\pi f)}e^{-i\pi fM}$
2. $\frac{\sin(2\pi Wn)}{\pi n}, \quad \lvert W\rvert < \frac{1}{2}$	
3. 	$\left(\frac{\sin(\pi f(M+1))}{\sin(\pi f)}\right)^2$
4. $a^n u(n), \quad \lvert a\rvert < 1$	$\frac{1}{1-ae^{-i2\pi f}}$
5. $a^{\lvert n\rvert}, \quad \lvert a\rvert < 1$	$\frac{1-a^2}{1-2a\cos(2\pi f)+a^2}$
6. $\delta(n)$	1
7. 1	$\delta(f)$
8. $\delta(n-M)$	$e^{-i2\pi fM}$
9. $e^{-i2\pi f_0 n}, \quad \lvert f_0\rvert < \frac{1}{2}$	$\delta(f-f_0)$
10. $\cos(2\pi f_0 n), \quad \lvert f_0\rvert < \frac{1}{2}$	$\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$
11. $\sin(2\pi f_0 n), \quad \lvert f_0\rvert < \frac{1}{2}$	$\frac{1}{2i}(\delta(f-f_0) - \delta(f+f_0))$
12. $u(n)$	$\frac{1}{2}\delta(f) + \frac{1}{1-e^{-i2\pi f}}$

...le J. Definitions & formulae

1. $X \sim \text{Poi}(\lambda),\ \mu = \lambda,\ \sigma^2 = \lambda,\ P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda},\ k = 0, 1, 2, \ldots$
2. $X \sim \text{Exp}(a),\ \mu = \frac{1}{a},\ \sigma^2 = \frac{1}{a^2},\ f(x) = ae^{-ax},\ x \geq 0$
3. $X \sim \text{N}(\mu, \sigma^2),\ f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
4. $X \sim \text{Rayleigh}(\alpha),\ \mu = \sqrt{\frac{\pi}{2\alpha}},\ \sigma^2 = \frac{4-\pi}{2\alpha},\ f(x) = \alpha x e^{-\frac{1}{2}\alpha x^2},\ x \geq 0$
5. $Z_k = \alpha Z_{k-1} \mod M$
6. $\{E[XY]\}^2 \leq E[X^2]E[Y^2]$
7. $C_Y = AC_X A^T$
8. $W(t) - W(s) \sim N(0, \sigma^2(t - s))$
9. $X(t) - X(s) \sim \text{Poi}[\lambda(t - s)]$
10. $X(t) = Y(t - A),\ A \sim \text{Tas}\ (0, \Delta),\ R_X(\tau) = 1 - |\tau|/\Delta,\ |\tau| \leq \Delta$
11. $X(t) = N(0, t),\ P(N(0, t) = k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$
12. $Z(t) = AY(t),\ R_Z(\tau) = e^{-2\lambda|\tau|}$
13. $E[(\int_a^b X(t)dt)^2] = \int_a^b\int_a^b R_X(t_1, t_2)dt_1 dt_2$
14. $S_X(f) = \lim_{T\to\infty} \frac{E[|\mathcal{X}_T(f)|^2]}{2T}, \qquad \langle R_X(t, t+\tau)\rangle = \int_{-\infty}^{\infty} S_X(f)e^{i2\pi f\tau}df$
15. $S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f\tau}d\tau, \qquad R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{i2\pi f\tau}df$
16. $S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-i2\pi fk}, \qquad R_X(k) = \int_{-1/2}^{1/2} S_X(f)e^{j2\pi fk}df$
17. $P_X = \int_{-\infty}^{\infty} S_X(f)df$
18. $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(s)X(t - s)ds = \int_{-\infty}^{\infty} h(t - s)X(s)ds$
19. $Y_n = h_n * X_n = \sum_{j=-\infty}^{\infty} h_j X_{n-j}$
20. $H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt$
21. $H(f) = \sum_{n=-\infty}^{\infty} h_n e^{-i2\pi fn}$
22. $S_Y(f) = |H(f)|^2 S_X(f)$
23. $R_{XY}(\tau) = R_X(\tau) * h(\tau)$
24. $S_{XY}(f) = H(f)S_X(f)$
25. $S_{YX}(f) = \overline{S_{XY}(f)} = \overline{H(f)}S_X(f)$
26. $H_{\text{opt}}(f) = \frac{1}{C}\frac{\overline{X(f)}}{S_N(f)}e^{-i2\pi ft_0}$
27. $H_{\text{opt}}(f) = \frac{S_{XZ}(f)}{S_X(f)},\ X(t) = Z(t) + N(t)$
28. $H_{\text{opt}}(f) = \frac{S_Z(f)}{S_Z(f)+S_N(f)},\ \epsilon = \int_{-\infty}^{\infty} \frac{S_Z(f)S_N(f)}{S_Z(f)+S_N(f)}df$
29. $Y_n = \sum_{k=a}^{b} h_k X_{n-k},\ Y(t) = \int_a^b h(s)X(t - s)ds$
30. $X_k = Z_k + N_k,\ X(t) = Z(t) + N(t)$
31. $R_{XZ}(m) = \sum_{k=a}^{b} h_k R_X(m - k),\ \epsilon = R_Z(0) - \sum_{k=a}^{b} h_k R_{XZ}(k)$
32. $R_{XZ}(\tau) = \int_a^b h(s)R_X(\tau - s)ds,\ \epsilon = R_Z(0) - \int_a^b h(s)R_{XZ}(s)ds$
33. $R_{XZ}(m) = \sum_{k=0}^{\infty} h_k R_X(m - k)$
34. $R_{XZ}(\tau) = \int_0^{\infty} h(s)R_X(\tau - s)ds$
35. $W(f) = 1/G(f),\ S_X(f) = G(f)\overline{G(f)}$
36. $H_2(f) = \sum_{m=0}^{\infty} R_{X'Z}(m)e^{-i2\pi fm}$
37. $R_{X'Z}(k) = \sum_{i=0}^{\infty} w_i R_{XZ}(k + i),\ S_{X'Z}(f) = S_{XZ}(f)/\overline{G(f)}$
38. $H(f) = W(f)H_2(f)$