Introduction to Optimization

Old exams, solutions

1.

$$x^{(0)} = [0\ 0]^\top, x^{(1)} = [-1/3\ 1/2]^\top \text{ and } x^{(2)} = [-0.3561, 0.5]^\top.$$

2. The function is quadratic since

$$f(x) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{b}^\top\boldsymbol{x}, \text{ where } A = \begin{bmatrix} 4 & -2 \\ -2 & 8 \end{bmatrix} \text{ and } \boldsymbol{b}^\top = [1\ 2].$$

Starting the conjugate gradient method from the initial point $x^{(0)} = [0\ 0]^\top$, we obtain $r^{(0)} = d_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\beta_0 = 0$, $\alpha_0 = 5/28$ and

$$x^{(1)} = \begin{bmatrix} 5/28 \\ 5/14 \end{bmatrix} \text{ and } r^{(1)} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}.$$

Next, $\beta_1 = 1/4$, $d_1 = \begin{bmatrix} 5/4 \\ 0 \end{bmatrix}$, $\alpha_1 = 1/5$ and

$$x^{(2)} = \begin{bmatrix} 3/7 \\ 5/14 \end{bmatrix} \text{ and } r^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $x^{(2)}$ is the optimal solution and the minimum value is $f\left(x^{(2)}\right) = -4/7$.

3.

$$x_{\min} = [9/7, -8/7]^\top$$

4. For the constrained problem

$$\min_{\substack{x_1 - x_2 - 2 \le 0 \\ x_1 - 2 \le 0}} \quad x_1^2 + x_1x_2 + 2x_2^2 - 2x_1$$

the Lagrangian is

$$L(x, \lambda) = x_1^2 + x_1x_2 + 2x_2^2 - 2x_1 + \lambda_1(x_1 - x_2 - 2) + \lambda_2(x_1 - 2).$$

The KKT-conditions are:

$$\begin{aligned}
&2x_1 + x_2 - 2 + \lambda_1 + \lambda_2 = 0, \\
&x_1 + 4x_2 - \lambda_1 = 0, \\
&\lambda_1 (x_1 - x_2 - 2) = 0, \\
&\lambda_2 (x_1 - 2) = 0, \\
&\lambda_1, \lambda_2 \ge 0.
\end{aligned}$$

When $\lambda_1 = \lambda_2 = 0$, $x = (8/7, -2/7) \in U$. This satisfies the conditions.
When $\lambda_1 = 0, \lambda_2 > 0$, we derive $\lambda_2 < 0$, which is not possible.
When $\lambda_1 > 0, \lambda_2 = 0$, $\lambda_1 < 0$ and the conditions are not satisfied.
When $\lambda_1 > 0, \lambda_2 > 0$, $\lambda_2 < 0$, which contradicts with the conditions.
The only point that satisfies the conditions is $[x, \lambda] = [8/7, -2/7, 0, 0]$.

5. The point $[x, u] = [0\ 1\ 1\ 0]^\top$.
6. For the primal problem

$$\min_{2x_1 + x_2 \le -2} 2x_1^2 + x_2^2 - x_1x_2 - x_2$$

the Lagrangian is

$$L(x, u) = 2x_1^2 + x_2^2 - x_1x_2 - x_2 + u\,(2x_1 + x_2 + 2)\,.$$

To find the dual function, we solve $\nabla_x L = 0$, that is,

$$\nabla_x L(x,u) = 0 \Leftrightarrow \begin{bmatrix} 4x_1 - x_2 + 2u \\ 2x_2 - x_1 - 1 + u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 = -\frac{5}{7}u + \frac{1}{7}, \\ x_2 = -\frac{6}{7}u + \frac{4}{7}. \end{cases}$$

Thus, the dual function is

$$G(u) = L\left(x_u, u\right) = L\left(-\tfrac{5}{7}u + \tfrac{1}{7}, -\tfrac{6}{7}u + \tfrac{4}{7}, u\right) = -\frac{8}{7}u^2 + \frac{20}{7}u - \frac{2}{7}$$

and the corresponding dual problem is

$$\max_{u \geq 0} G(u).$$

7. First $x_1 = \frac{1}{7}(2 - 5u)$ and $x_2 = \frac{1}{7}(1 - 6u)$. Substituting these to Lagrangian the dual function is found: $G(u) = -\frac{8}{7}u^2 + \frac{12}{7}u - \frac{1}{7}$. The dual problem is

$$\max_{u \geq 0} G(u).$$

The maximum is found at $u_{\max} = \frac{3}{4}$. Then $x_1 = -\frac{1}{4}$ and $x_2 = -\frac{1}{2}$.

8. In the constrained optimization problem

$$\min_{-x_1 + 2x_2 - 2 \leq 0} x_1^2 + x_2^2 + x_1 x_2 - 3x_2$$

the objective function is $f = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{b}^\top \boldsymbol{x}$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $b^\top = [0\ 3]$. The constraint set is $U = \{x : C\boldsymbol{x} - d\}$, where $C = [-1\ 2]$ and $d = 2$. The initial dual variable $\lambda^{(0)} = 1$ and the step size $\rho = \frac{1}{7}$. Then

$$x^{(0)} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}, \ \lambda^{(1)} = \tfrac{1}{6}21, \ x^{(1)} = \begin{bmatrix} 1/63 \\ 46/63 \end{bmatrix}, \ \lambda^{(2)} = \tfrac{4}{3}63, \ x^{(2)} = \begin{bmatrix} -17/189 \\ 163/189 \end{bmatrix}.$$