Introduction to Optimization

Fall 2015, Homework 3

5. Let $X = (x_{ij})$, $i, j = 1, \dots, n$ be a symmetric and positive definite matrix. We minimize

$$f(X) = \log\left(\det\left(X^{-1}\right)\right)$$

subject to

$$\boldsymbol{a}_i^\top X \boldsymbol{a}_i \leq 1, \; i = 1, \dots, m,$$

where the points $\boldsymbol{a}_i$, $i = 1, \dots, m$ span the space $\mathbb{R}^n$.
To find the dual function, we need to find the minimum of the Lagrangian

$$L(x, u) = \log\left(\det\left(X^{-1}\right)\right) + \sum_{i=1}^{m} u_i \left(\boldsymbol{a}_i^\top X \boldsymbol{a}_i - 1\right).$$

Since $\det\left(X^{-1}\right) = \det\left(X\right)^{-1}$, we obtain

$$\log\left(\det\left(X^{-1}\right)\right) = \log\left(\det\left(X\right)^{-1}\right) = -\log\left(\det X\right).$$

Moreover,

$$\begin{aligned} \frac{\partial}{\partial x_{ij}} \log\left(\det\left(X^{-1}\right)\right) &= -\frac{\partial}{\partial x_{ij}} \log\left(\det(X)\right) = -\frac{1}{\det(X)} \frac{\partial}{\partial x_{ij}} \det(X) \\ &= -\frac{1}{\det(X)} \operatorname{adj}\left(X\right)_{ji} = -\left(X^{-1}\right)_{ji} \end{aligned}$$

and now

$$\begin{aligned} \nabla_x L(x, u) = 0 &\Leftrightarrow \left(X^{-1}\right)_{ji}^\top = \sum_{k=1}^{m} u_k a_{ki} a_{kj} \\ &\Leftrightarrow \left(X^{-1}\right)_{ji} = u^\top \begin{bmatrix} a_{1i}a_{1j} \\ a_{2i}a_{2j} \\ \vdots \\ a_{mi}a_{mj} \end{bmatrix}. \end{aligned}$$

Therefore,

$$\left(X^{-1}\right)^\top = AUA^\top,$$

where $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_m \end{bmatrix}$ and $U = \begin{bmatrix} u_1 & 0 & 0 & \dots & 0 \\ 0 & u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_m \end{bmatrix}$.

Hence,

$$X = \left(A^{-1}\right)^\top U^{-1} A^{-1}.$$

Thus, the dual function is

$$G(u) = \log\left(\det\left(AUA^\top\right)\right) + \sum_{i=1}^{m} u_i \left(\boldsymbol{a}_i^\top X \boldsymbol{a}_i - 1\right), \tag{0.1}$$

$$\text{where } \; X = \left(A^{-1}\right)^\top U^{-1} A^{-1} \text{ and } u \geq 0.$$

The dual problem is

$$\max_{u \geq 0} \; G(u).$$

Let now $\boldsymbol{a}_1 = [1\ 0]^\top$ and $\boldsymbol{a}_2 = [1\ 1]^\top$. Then $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the inverse is $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{aligned} G(u) &= \log\left(\det\left(AUA^{-1}\right)\right) + u_1 \left(\boldsymbol{a}_1^\top \left(A^{-1}\right)^\top U^{-1} A^{-1} \boldsymbol{a}_1 - 1\right) \\ &\quad + u_2 \left(\boldsymbol{a}_2^\top \left(A^{-1}\right)^\top U^{-1} A^{-1} \boldsymbol{a}_2 - 1\right) \\ &= \log(u_1 u_2) + u_1 \left([1\ 0] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{-1} & 0 \\ 0 & u_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1\right) \\ &\quad + u_2 \left([1\ 1] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{-1} & 0 \\ 0 & u_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\right) \\ &= \log(u_1 u_2) - u_1 - u_2 + 2 \end{aligned}$$

and

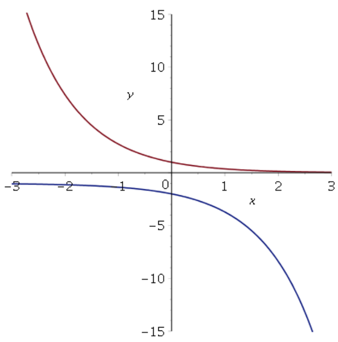
$$\nabla G(u) = 0 \Leftrightarrow \begin{cases} \frac{1}{u_1} - 1 = 0 \\ \frac{1}{u_2} - 1 = 0 \end{cases} \Leftrightarrow u_1 = u_2 = 1.$$

Thus,

$$X = \left(A^{-1}\right)^\top U^{-1} A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } V = C_2 \sqrt{\det\left(X^{-1}\right)} = \pi.$$

6. The smallest distance between the sets is obtained on the boundary of the sets U_0 and U_1. Thus we set $x_2 = \mathrm{e} x_1$ and solve the following problem

$$\min_{y \in U_1} \|(x_1, x_2) - (y_1, y_2)\|^2 = \min_{y \in U_1} (x_1 - y_1)^2 + (\mathrm{e}^{-x_1} - y_2)^2.$$



The Lagrangian functional is

$$L(x, \lambda) = (x_1 - y_1)^2 + (\mathrm{e}^{-x_1} - y_2)^2 + \lambda\,(1 + \mathrm{e}^{y_1} + y_2)\,.$$

Hence

$$\nabla_x L(x, \lambda) = 0 \Leftrightarrow \begin{cases} 2(x_1 - y_1) - 2\lambda\,\mathrm{e}^{-x_1}\,(\mathrm{e}^{-x_1} - y_2) = 0, \\ -2(x_1 - y_1) + \lambda\,\mathrm{e}^{y_1} = 0, \\ -2(\mathrm{e}^{-x_1} - y_2) + \lambda = 0. \end{cases}$$

We set $\lambda^0 = 1$ and solve $\nabla_x L(x, 1) = 0$. That is, $x_1^0 = 0.2039,\ y_2 = -0.2039,\ y_2 = 0.3156$ and

$$\lambda^1 = \max\left\{1 + \rho\,(1 + \mathrm{e}^{y_1} + y_2)\,, 0\right\} \overset{\rho=1}{=} 3.1311.$$

Now $\nabla_x L(x, \lambda^1) = 0 \Leftrightarrow x_1^1 = 0.4829, y_1^1 = -0.4829, y_2^1 = -0.9486$. The following steps are:

- $\lambda^2 = 3.7995,\ x_1^2 = 0.5487,\ y_1^2 = -0.5487,\ y_2^2 = -1.3221$
- $\lambda^3 = 4.0551,\ x_1^3 = 0.5721,\ y_1^3 = -0.5721,\ y_2^3 = -1.4632$
- $\lambda^4 = 4.1562,\ x_1^4 = 0.5811,\ y_1^4 = -0.5811,\ y_2^4 = -1.5188$
- ...

The minimum point is attained at $(x_1, x_2, y_1, y_2) = (0.5871, 0.5560, -0.5871, -1.5560)$ and the minimum distance is 2.4164.