Introduction to Optimization

Fall 2015, Homework 2

3. The function $J(x_1, x_2, x_3) = 4x_1^2 + 4x_2^2 + 4x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_1 + 2x_2 + 2x_3$ can be written in a quadratic form

$$J(x_1, x_2, x_3) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{b}^\top\boldsymbol{x}, \text{ where } A = \begin{bmatrix} 8 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 8 \end{bmatrix} \text{ and } \boldsymbol{b}^\top = [2 \ -2 \ -2].$$

The initial point is $\boldsymbol{x}^{(0)} = [0 \ 0 \ 0]^\top$ and $\boldsymbol{r}^{(0)} = d_0 = \boldsymbol{b} = [2 \ -2 \ -2]^\top$. Moreover,

$$\beta_0 = 0, \ \alpha_0 = \frac{\left\|\boldsymbol{r}^{(0)}\right\|^2}{\boldsymbol{d}_0^\top A\boldsymbol{d}_0} = \frac{1}{8} \text{ and thus } \boldsymbol{x}^{(1)} = \alpha_0\boldsymbol{d}_0 = \begin{bmatrix} 1/4 \\ -1/4 \\ -1/4 \end{bmatrix}.$$

Furthermore,

$$\boldsymbol{r}^{(1)} = \boldsymbol{r}^{(0)} - \alpha_0 A\boldsymbol{d}_0 = \begin{bmatrix} -1/2 \\ 0 \\ -1/2 \end{bmatrix}, \ \beta_1 = \frac{\left\|\boldsymbol{r}^{(1)}\right\|^2}{\left\|\boldsymbol{r}^{(0)}\right\|^2} = \frac{1}{24}, \ \boldsymbol{d}_1 = \boldsymbol{r}^{(1)} + \beta_1\boldsymbol{d}_0 = \begin{bmatrix} -5/12 \\ -1/12 \\ -7/12 \end{bmatrix}$$

and $\alpha_1 = \frac{\left\|\boldsymbol{r}^{(1)}\right\|^2}{\boldsymbol{d}_1^\top A\boldsymbol{d}_1} = \frac{3}{23}$. Therefore

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 9/46 \\ -6/23 \\ -15/46 \end{bmatrix}$$

and $\boldsymbol{r}^{(2)} = \begin{bmatrix} -2/23 \\ -4/23 \\ 2/23 \end{bmatrix}$, $\beta_2 = \frac{48}{529}$, $\boldsymbol{d}_2 = \begin{bmatrix} -66/529 \\ -96/529 \\ 18/529 \end{bmatrix}$ and $\alpha_2 = \frac{23}{168}$.

Hence,

$$\boldsymbol{x}^{(3)} = \begin{bmatrix} 5/28 \\ -2/7 \\ -9/28 \end{bmatrix}.$$

The residual $\boldsymbol{r}^{(3)} = 0$, and thus the optimal point is $\boldsymbol{x}^{(3)}$ with the value $J\left(\boldsymbol{x}^{(3)}\right) = -\frac{11}{14} \approx -0,786$.

4. We minimize a function $f(x_1, x_2, x_3) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x}$ under the constraint set $U = \{\boldsymbol{x} : \boldsymbol{x}^\top\boldsymbol{x} - 1 \leq 0\}$.
Here $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and $\boldsymbol{b}^\top = [1 \;\; -1 \;\; 0]$. First, the matrix A is not positive definite, since

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1/2 \end{bmatrix}.$$

Thus, f is not convex and the KKT-conditions are only necessary, not sufficient.
The Lagrangian is

$$L(x, u) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{b}^\top\boldsymbol{x} + u\boldsymbol{x}^\top\boldsymbol{x}.$$

The (KKT1) condition yields to $\nabla_x L(x, u) = A\boldsymbol{x} + \boldsymbol{b} + 2u\boldsymbol{x} = 0$. That is,

$$\boldsymbol{x} = -(A + 2uI)^{-1}\boldsymbol{b}.$$

From the conditions (KKT2) and (KKT3) we obtain

$$u(\boldsymbol{x}^\top\boldsymbol{x} - 1) = 0 \text{ where } u \geq 0.$$

To find $\boldsymbol{x}$ we compute the inverse of $A + 2uI$. The determinant is

$$\det(A + 2uI) = \begin{vmatrix} 1+2u & 0 & 1 \\ 0 & 2+2u & -1 \\ 1 & -1 & 1+2u \end{vmatrix} = 8u^3 + 16u^2 + 6u - 1 = 0.$$

Thus, the inverse exists whenever $8u^3 + 16^2 + 6u - 1 \neq 0$.

Assumption $u = 0$ yields to $\boldsymbol{x} = -A^{-1}\boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \notin U$.

Let $u \neq 0$. Then

$$(A + 2uI)^{-1} = \frac{1}{8u^3 + 16u^2 + 6u - 1} \begin{bmatrix} 4u^2 + 6u + 1 & -1 & -2u - 2 \\ -1 & 4u^2 + 4u & 2u + 1 \\ -2u - 2 & 2u + 1 & 4u^2 + 6u + 2 \end{bmatrix}$$

and

$$\begin{aligned} \boldsymbol{x} &= -\frac{1}{8u^3 + 16u^2 + 6u - 1} \begin{bmatrix} 4u^2 + 6u + 1 & -1 & -2u - 2 \\ -1 & 4u^2 + 4u & 2u + 1 \\ -2u - 2 & 2u + 1 & 4u^2 + 6u + 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ &= -\frac{1}{8u^3 + 16u^2 + 6u - 1} \begin{bmatrix} 4u^2 + 6u + 2 \\ -4u^2 - 4u - 1 \\ -4u - 3 \end{bmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} \boldsymbol{x}^\top\boldsymbol{x} =& 1 \\ \Leftrightarrow& \frac{1}{(8u^3 + 16u^2 + 6u - 1)^2}(32u^4 + 80u^3 + 92u^2 + 56u + 14) = 1 \\ \Leftrightarrow& u \approx 0{,}60094 \text{ (the rest of the roots do not satisfy (KKT3))}. \end{aligned}$$

Thus, the point x is:

$$\boldsymbol{x} = -\frac{1}{8u^3 + 16u^2 + 6u - 1} \begin{bmatrix} 4u^2 + 6u + 1 & -1 & -2u - 2 \\ -1 & 4u^2 + 4u & 2u + 1 \\ -2u - 2 & 2u + 1 & 4u^2 + 6u + 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} -0.6967 \\ 0.4791 \\ 0.5340 \end{bmatrix} \in U.$$

This is the minimum point and $f(x_{opt}) \approx -1.19$.