Faculty of technology, Mathematics division

Introduction to optimization

Homeworks, fall 2015

5. An ellipsoid, centered at the origin, in the Euclidean space $\mathbb{R}^n$ is determined by a positive definite matrix $X = (x_{i,j}),\ i, j = 1, \dots, n$. It contains the points $\mathbf{y}$ for which

$$\mathbf{y}^\top X \mathbf{y} \leq 1.$$

It is known that the volume of the ellipsoid $V_n(X)$ is

$$V_n = C_n \sqrt{\det(X^{-1})},$$

where C_n is the dimension dependent constant of proportionality, i.e. the volume of the unit ball in $\mathbb{R}^n$.

Let then $\mathbf{a}_i,\ i = 1, \dots, m$, be a set of points that spans the Euclidean space $\mathbb{R}^n$ $(m \geq n)$. The problem is to find the minimum volume ellipsoid that encloses the given points. Instead of finding the minimum volume, it is easier (I think) to minimize the logarithm of the volume. Hence the problem can be written as to

$$\begin{aligned} &\text{minimize } f(X) = \log(\det(X^{-1})) \\ &\text{subject to } \mathbf{a}_i^\top X \mathbf{a}_i \leq 1,\ i = 1, \dots, m. \end{aligned}$$

a) Find the dual function $G(u)$ and the corresponding dual problem.

b) If $\mathbf{a}_1 = [1,\ 0]^\top$, $\mathbf{a}_2 = [1,\ 1]^\top$, find the smallest origo-centered ellipse that contains these points.

6. The sets

$$\begin{aligned} U_0 &= \{x :\ e^{-x_1} - x_2 \leq 0\} \\ U_1 &= \{y :\ x_2 + e^{x_1} + 1 \leq 0\} \end{aligned}$$

are convex. Compute the distance between the sets using the Uzawa's method. The distance of the two sets is defined as

$$\min_{\substack{x \in U_0 \\ y \in U_1}} \|x - y\|.$$

Return the homework to Optima before 4.00 pm 20.10.2015