Introduction to Optimization

Final exam 22.10.2015

1. The gradient and the Hessian matrix for $f(x) = e^{-x_1-x_2} + x_1^2 + x_2^2$ are

$$\nabla f(x) = \begin{bmatrix} 2x_1 - e^{-x_1-x_2} \\ 2x_2 - e^{-x_1-x_2} \end{bmatrix} \text{ and } H_f = \begin{bmatrix} 2 + e^{-x_1-x_2} & e^{-x_1-x_2} \\ e^{-x_1-x_2} & 2 + e^{-x_1-x_2} \end{bmatrix}.$$

The Hessian matrix is positive definite for every $x \in \mathbb{R}^2$, and therefore f is strictly convex. The Newton iterations are:

$$x^{(k+1)} = x^{(k)} - \left[H_f\left(x^{(k)}\right)\right]^{-1} \nabla f\left(x^{(k)}\right).$$

The starting point is $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Next iteration is

$$x^{(1)} = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} \quad \left(\text{and } \nabla f\left(x^{(1)}\right) \approx \begin{bmatrix} -0,1065 \\ -0,1065 \end{bmatrix}\right).$$

Moreover,

$$x^{(2)} = \begin{bmatrix} 0,2832 \\ 0,2832 \end{bmatrix} \quad \left(\text{and } \nabla f\left(x^{(2)}\right) \approx \begin{bmatrix} -0,0013 \\ -0,0013 \end{bmatrix}\right).$$

2. The function is quadratic since

$$f(x) = \frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{b}^\top\boldsymbol{x}, \text{ where } A = \begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix} \text{ and } \boldsymbol{b}^\top = [-2 \ -3].$$

Starting the conjugate gradient method from the initial point $x^{(0)} = [0\ 0]^\top$, we obtain $r^{(0)} = d_0 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\beta_0 = 0$, $\alpha_0 = 13/44$ and

$$x^{(1)} = \frac{13}{44}\begin{bmatrix} -2 \\ -3 \end{bmatrix} \text{ and } r^{(1)} = \frac{7}{22}\begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Next, $\beta_1 = \frac{7^2}{22^2}$, $d_1 = \frac{7\cdot 13}{22^2}\begin{bmatrix} 4 \\ -5 \end{bmatrix}$, $\alpha_1 = \frac{11}{7\cdot 13}$ and

$$x^{(2)} = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} \text{ and } r^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $x^{(2)}$ is the optimal solution.

3. From the KKT-conditions we obtain

$$\begin{aligned} e^{x_1-x_2} + u_1 e^{x_1} - u_2 &= 0, \\ -e^{x_1-x_2} + u_1 e^{x_2} &= 0, \\ u_1 (e^{x_1} + e^{x_2} - 20) &= 0, \\ u_2 x_1 &= 0, \\ u_1, u_2 &\geq 0. \end{aligned}$$

The conditions are satisfied when $u_1 \neq 0$ and $u_2 \neq 0$ (all other possibilities lead to contradiction). Since $u_2 \neq 0$, we must have $x_1 = 0$ and $e^{x_1} + e^{x_2} - 20 = 0$. Thus, $e^{x_2} = 19$ and therefore $x_2 = \ln 19$. Moreover, $u_1 = \frac{1}{361}$ and $u_2 = \frac{20}{361}$.

4. The Lagrangian is

$$L(x,u) = x_1^2 + x_1x_2 + x_2^2 - x_1 - x_2 + u_1(x_1 - 1) + u_2(-x_1 - 1) + u_3(x_2 - 1) + u_4(-x_2 - 1).$$

The gradient is

$$\nabla L(x,u) = \begin{bmatrix} 2x_1 + x_2 - 1 + u_1 - u_2 \\ x_1 + 2x_2 - 1 + u_3 - u_4 \end{bmatrix}$$

and setting the gradient to zero and solving x_1 and x_2 with respect to u's, one obtain

$$x_1 = \frac{1}{3}(1 - 2u_1 + 2u_2 + u_3 - u_4) \text{ and } x_2 = \frac{1}{3}(1 + u_1 - u_2 - 2u_3 + 2u_4).$$

Substituting these to the Lagrangian we obtain the dual function

$$\begin{aligned} G(u) &= -(1/3)u_3^2 + (2/3)u_3u_4 + (1/3)u_3u_1 - (1/3)u_3u_2 - (1/3)u_4^2 - (1/3)u_4u_1 + (1/3)u_4u_2 \\ &- (1/3)u_1^2 + (2/3)u_1u_2 - (1/3)u_2^2 - 1/3 - (2/3)u_3 - (2/3)u_1 - (4/3)u_4 - (4/3)u_2. \end{aligned}$$

The roots of the gradient of the dual function do not exist, and therefore the global solution $x_{opt} = [\frac{1}{3}, \frac{1}{3}]^\top$ is the solution.