MATEMATIIKAN PERUSKURSSI II, kevät 2024

Harjoitus 8 ratkaisut

1. Kappale voidaan kuvata seuraavasti:

$$V = \{(x,y,z) \in \mathbb{R}^3 :\ 0 \le x \le 2, 0 \le y \le 2-x, -1 \le z \le 4-x^2-y^2\}.$$

Lasketaan kappaleen tilavuus:

$$\begin{aligned}
\iiint\limits_V 1\,\mathrm{d}V &= \int_0^2 \int_0^{2-x} \int_{-1}^{4-x^2-y^2} \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^2 \int_0^{2-x} \Big(\Big/_{-1}^{4-x^2-y^2} z \Big)\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^2 \int_0^{2-x} (5-x^2-y^2)\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^2 \Big/_0^{2-x} (5y - x^2y - \frac{1}{3}y^3)\ \mathrm{d}x \\
&= \int_0^2 [5(2-x) - x^2(2-x) - \frac{1}{3}(2-x)^3]\ \mathrm{d}x \\
&= \int_0^2 [10 - 5x - 2x^2 + x^3 - \frac{1}{3}(2-x)^3]\ \mathrm{d}x \\
&= \Big/_0^2 [10x - \frac{5}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{12}(2-x)^4] \\
&= \frac{22}{3} = 7\frac{1}{3}.
\end{aligned}$$

2. a) Alue $A = \{(x,y) \in \mathbb{R}^2 :\ -3 \le x \le 0, 0 \le y \le \sqrt{9-x^2}\}$ voidaan kuvata napakoordinaattien avulla muodossa

$$A' = \{(r,\varphi) \in \mathbb{R}^2 :\ 0 \le r \le 3, \frac{\pi}{2} \le \varphi \le \pi\}.$$

$$\begin{aligned}
&\int_{-3}^{0} \int_0^{\sqrt{9-x^2}} \mathrm{e}^{25-x^2-y^2}\ \mathrm{d}y\ \mathrm{d}x \\
&= \iint\limits_A \mathrm{e}^{25-x^2-y^2}\,\mathrm{d}A = \iint\limits_{A'} \mathrm{e}^{25-r^2}\ r\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_{\frac{\pi}{2}}^{\pi} \int_0^3 r\,\mathrm{e}^{25-r^2}\ \mathrm{d}r\,\mathrm{d}\varphi = -\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \int_0^3 (-2r)\,\mathrm{e}^{25-r^2}\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= -\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \Big/_0^3 \mathrm{e}^{25-r^2}\,\mathrm{d}\varphi = -\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\mathrm{e}^{16} - \mathrm{e}^{25})\,\mathrm{d}\varphi \\
&= \frac{\pi}{4}(\mathrm{e}^{25} - \mathrm{e}^{16})
\end{aligned}$$

b) Alue $A = \{(x,y) \in \mathbb{R}^2 :\ 9 \leq x^2+y^2 \leq 25, x \leq 0, y \geq 0\}$ voidaan kuvata napakoordinaattien avulla muodossa

$$A' = \{(r,\varphi) \in \mathbb{R}^2 :\ 3 \leq r \leq 5, \frac{\pi}{2} \leq \varphi \leq \pi\}.$$

$$\begin{aligned}
\iint\limits_A \frac{\sqrt{x^2+y^2}}{(x^2+y^2)\sqrt{x^2+y^2+2}}\,\mathrm{d}A &= \iint\limits_{A'} \frac{\sqrt{r^2}}{r^2\sqrt{r^2+2}}\, r\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_{\frac{\pi}{2}}^{\pi}\int_3^5 \frac{r^2}{r^3+2}\ \mathrm{d}r\,\mathrm{d}\varphi = \frac{1}{3}\int_{\frac{\pi}{2}}^{\pi}\int_3^5 \frac{3r^2}{r^3+2}\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \frac{1}{3}\int_{\frac{\pi}{2}}^{\pi} \Big/_3^5 \ln|r^3+2|\,\mathrm{d}\varphi = \frac{1}{3}\int_{\frac{\pi}{2}}^{\pi}[\ln(127)-\ln(29)]\,\mathrm{d}\varphi \\
&= \frac{\pi}{6}\ln\frac{127}{29}
\end{aligned}$$

3. Tarvitaan sylinterikoordinaatit

$$\begin{cases} x &= r\cos\varphi, \\ y &= r\sin\varphi, \\ z &= z. \end{cases}$$

Kappale voidaan kuvata sylinterikoordinaattien avulla seuraavasti:

$$V' = \{(r,\varphi,z) \in \mathbb{R}^3 :\ 0 \leq r \leq 2, 0 \leq \varphi \leq 2\pi, 0 \leq z \leq \sqrt{9-r^2}\}.$$

Lasketaan kappaleen tilavuus:

$$\begin{aligned}
\iiint\limits_V 1\,\mathrm{d}V &= \iiint\limits_{V'} r\ \mathrm{d}r\,\mathrm{d}\varphi\ \mathrm{d}z = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-r^2}} r\ \mathrm{d}z\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_0^{2\pi}\int_0^2 \Big(\Big/_0^{\sqrt{9-r^2}} rz\Big)\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^2 r\sqrt{9-r^2}\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_0^{2\pi}\int_0^2 -\frac{1}{2}(-2r)(9-r^2)^{\frac{1}{2}}\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_0^{2\pi}\Big[\Big/_0^2 -\frac{1}{3}(9-r^2)^{\frac{3}{2}}\Big]\,\mathrm{d}\varphi = \int_0^{2\pi} -\frac{1}{3}(5^{\frac{3}{2}}-9^{\frac{3}{2}})\,\mathrm{d}\varphi \\
&= \frac{2\pi}{3}(27-5\sqrt{5}).
\end{aligned}$$

4. Kappale voidaan kuvata sylinterikoordinaateissa seuraavasti:

$$V' = \{(r,\varphi,z) \in \mathbb{R}^3 :\ 0 \leq r \leq 1, 0 \leq \varphi \leq \frac{\pi}{2}, r \leq z \leq \sqrt{2-r^2}\}.$$

Tällöin

$$\begin{aligned}
\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^3\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x &= \int_0^{\frac{\pi}{2}}\int_0^1\int_r^{\sqrt{2-r^2}} z^3\, r\ \mathrm{d}z\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{\frac{\pi}{2}}\int_0^1\Big(\Big/_r^{\sqrt{2-r^2}} \frac{rz^4}{4}\Big)\ \mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}}\int_0^1 (r-r^3)\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{\frac{\pi}{2}}\Big/_0^1 \Big(\frac{1}{2}r^2-\frac{1}{4}r^4\Big)\,\mathrm{d}\varphi = \frac{\pi}{8}.
\end{aligned}$$

5. a) Tarvitaan pallokoordinaatit

$$\begin{cases} x &= \rho \sin\theta \cos\varphi, \\ y &= \rho \sin\theta \sin\varphi, \\ z &= \rho \cos\theta. \end{cases}$$

Annettu kappale pallokoordinaateissa on

$$V' = \{(\rho, \theta, \varphi) \in \mathbb{R}^3 : \ 0 \le \rho \le 4, \frac{\pi}{2} \le \theta \le \pi, 0 \le \varphi \le \frac{\pi}{2}\}.$$

Lasketaan kysytty integraali:

$$\begin{aligned} \iiint\limits_V \frac{\mathrm{e}^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}V &= \iiint\limits_{V'} \frac{\mathrm{e}^{\rho^2}}{\rho}\,\rho^2 \sin\theta \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi = \int\limits_0^{\frac{\pi}{2}} \int\limits_{\frac{\pi}{2}}^{\pi} \int\limits_0^4 \rho\,\mathrm{e}^{\rho^2} \sin\theta \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int\limits_0^{\frac{\pi}{2}} \int\limits_{\frac{\pi}{2}}^{\pi} \Big(\Big/_{\!0}^{4} \frac{1}{2}\,\mathrm{e}^{\rho^2}\Big) \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi = \int\limits_0^{\frac{\pi}{2}} \int\limits_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(\,\mathrm{e}^{16}-1) \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int\limits_0^{\frac{\pi}{2}} \Big/_{\!\frac{\pi}{2}}^{\pi} \frac{1}{2}(\,\mathrm{e}^{16}-1)(-\cos\theta)\,\mathrm{d}\varphi = \frac{\pi}{4}(\,\mathrm{e}^{16}-1). \end{aligned}$$

b) Annettu kappale pallokoordinaateissa on

$$V' = \{(\rho, \theta, \varphi) \in \mathbb{R}^3 : \ \sqrt{R} \le \rho \le 2\sqrt{R}, 0 \le \theta \le \frac{\pi}{2}, \pi \le \varphi \le \frac{3\pi}{2}\}.$$

Lasketaan kysytty integraali

$$\begin{aligned} I_R &= \iiint_V \frac{z}{(x^2+y^2+z^2)^2+1}\,\mathrm{d}V = \iiint_{V'} \frac{\rho\cos\theta}{(\rho^2)^2+1}\,\rho^2 \sin\theta \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int\limits_{\pi}^{\frac{3\pi}{2}} \int\limits_0^{\frac{\pi}{2}} \int\limits_{\sqrt{R}}^{2\sqrt{R}} \frac{\rho^3}{\rho^4+1} (\cos\theta\sin\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int\limits_{\pi}^{\frac{3\pi}{2}} \int\limits_0^{\frac{\pi}{2}} \Big/_{\!\sqrt{R}}^{2\sqrt{R}} \frac{1}{4}\ln(\rho^4+1)(\cos\theta\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int\limits_{\pi}^{\frac{3\pi}{2}} \int\limits_0^{\frac{\pi}{2}} \frac{1}{4}[\ln(16R^2+1)-\ln(R^2+1)](\cos\theta\sin^1\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \frac{1}{4}\ln\frac{16R^2+1}{R^2+1} \int\limits_{\pi}^{\frac{3\pi}{2}} \Big/_{\!0}^{\frac{\pi}{2}} \frac{\sin^2\theta}{2}\,\mathrm{d}\varphi \\ &= \frac{\pi}{16}\ln\frac{16R^2+1}{R^2+1}. \end{aligned}$$

Luvun $R > 0$ ratkaiseminen:

$$\frac{\pi}{16}\ln\frac{16R^2+1}{R^2+1} = \frac{\pi}{16}\ln 7 \quad\Rightarrow\quad \frac{16R^2+1}{R^2+1} = 7 \quad\Rightarrow\quad R = \pm\frac{\sqrt{6}}{3},$$

josta saadaan $R = \frac{\sqrt{6}}{3}$.