MATEMATIIKAN PERUSKURSSI II, kevät 2024

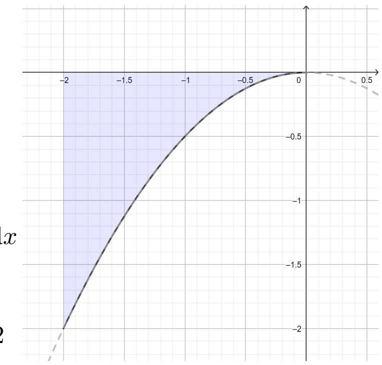
Harjoitus 7 ratkaisut

1. a)

$$\int_{-2}^{0}\int_{-\frac{1}{2}x^2}^{0} 14x^2y \, \mathrm{d}y \, \mathrm{d}x = \int_{-2}^{0}\left(\Big/_{-\frac{1}{2}x^2}^{0} 7x^2y^2 \right) \mathrm{d}x$$

$$= \int_{-2}^{0} 7x^2(0 - \left(-\frac{x^2}{2}\right)^2) \, \mathrm{d}x = \int_{-2}^{0} (-\frac{7}{4}x^6) \, \mathrm{d}x$$

$$= \Big/_{-2}^{0} -\frac{1}{4}(0-(-2)^7) = -\frac{2^7}{2^2} = -2^5 = -32$$

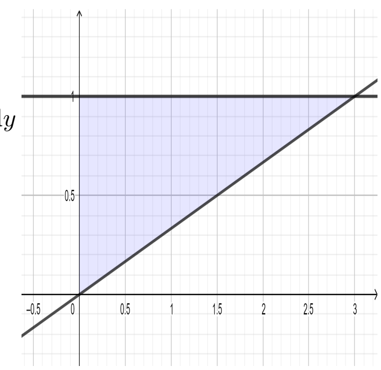


b)

$$\int_{0}^{1}\int_{0}^{3y} x\sqrt{9y^2-x^2} \, \mathrm{d}x \, \mathrm{d}y = \int_{0}^{1}\left(\Big/_{0}^{3y} -\frac{1}{3}\left(9y^2-x^2\right)^{\frac{3}{2}} \right) \mathrm{d}y$$

$$= \int_{0}^{1} \frac{1}{3}(9y^2)^{\frac{3}{2}} \, \mathrm{d}y = \int_{0}^{1} 9y^3 \, \mathrm{d}y$$

$$= \Big/_{0}^{1} \frac{9}{4}y^4 = \frac{9}{4}$$



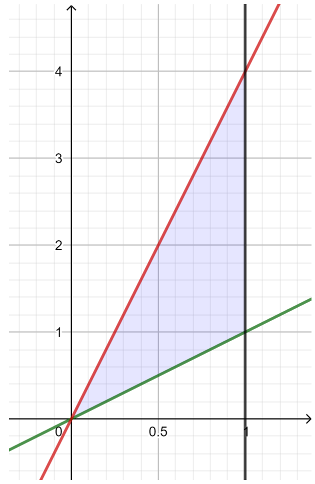
2. a) Alue voidaan kuvata muodossa $A = \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, x \le y \le 4x\}$ ja

$$\iint_A 2y(x^3+1)^4 \mathrm{d}A = \int_{0}^{1}\int_{x}^{4x} 2y(x^3+1)^4 \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_{0}^{1}\left(\Big/_{x}^{4x} y^2(x^3+1)^4 \right) \mathrm{d}y$$

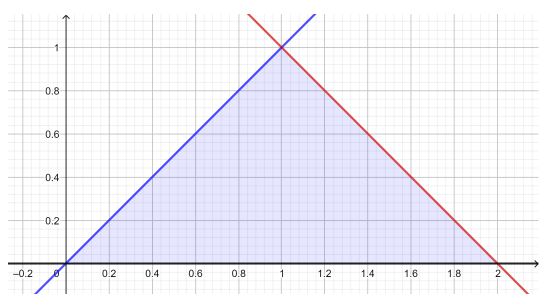
$$= \int_{0}^{1} (15x^2(x^3+1)^4) \, \mathrm{d}x$$

$$= \Big/_{0}^{1} (x^3+1)^5 = 2^5 - 1^5 = 31.$$



b) Alue voidaan kuvata muodossa $A = \big\{(x,y) \in \mathbb{R}^2 : \ 0 \le y \le 1, y \le x \le 2-y\big\}$ ja

$$\begin{aligned}
\iint\limits_A \frac{2}{-y^2+2y+4}\,\mathrm{d}A &= \int_0^1 \int_y^{2-y} \frac{2}{-y^2+2y+4}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left(\Big/_{y}^{2-y} \frac{2x}{-y^2+2y+4} \right) \mathrm{d}y \\
&= \int_0^1 \frac{2(-2y+2)}{-y^2+2y+4}\,\mathrm{d}y \\
&= \Big/_{0}^{1}\, 2\ln|-y^2+2y+4| = 2(\ln(5)-\ln(4)) = 2\ln\left(\frac{5}{4}\right).
\end{aligned}$$



3. a)

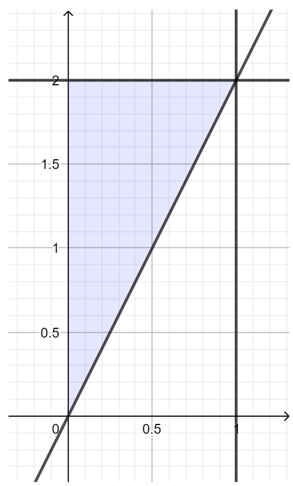
$$\int_0^1 \int_{2x}^2 8x\sqrt{y^3+1}\ \mathrm{d}y\ \mathrm{d}x = \int_0^2 \int_0^{\frac{y}{2}} 8x\sqrt{y^3+1}\ \mathrm{d}x\ \mathrm{d}y$$

$$= \int_0^2 \left(\Big/_0^{\frac{y}{2}} 4x^2\sqrt{y^3+1} \right)\ \mathrm{d}y$$

$$= \int_0^2 y^2\sqrt{y^3+1}\ \mathrm{d}y$$

$$= \Big/_0^2 \frac{1}{3}(y^3+1)^{\frac{3}{2}}$$

$$= \frac{2}{9}\left(3\sqrt{3}-1\right)$$



b)

$$\int_0^4 \int_{M\sqrt{y}}^{2M} \frac{2}{\sqrt[3]{x^3+M^3}}\ \mathrm{d}x\ \mathrm{d}y = \int_0^{2M} \int_0^{\frac{x^2}{M^2}} \frac{2}{\sqrt[3]{x^3+M^3}}\ \mathrm{d}y\ \mathrm{d}x$$

$$= \int_0^{2M} \left(\Big/_0^{\frac{x^2}{M^2}} 2(x^3+M^3)^{-\frac{1}{3}}y \right)\ \mathrm{d}x = \int_0^{2M} \frac{2x^2}{M^2}(x^3+M^3)^{-\frac{1}{3}}\ \mathrm{d}x$$

$$= \Big/_0^{2M} \frac{1}{M^2}(x^3+M^3)^{\frac{2}{3}} = \frac{1}{M^2}((9M^3)^{\frac{2}{3}} - (M^3)^{\frac{2}{3}})$$

$$= 3\sqrt[3]{3} - 1$$

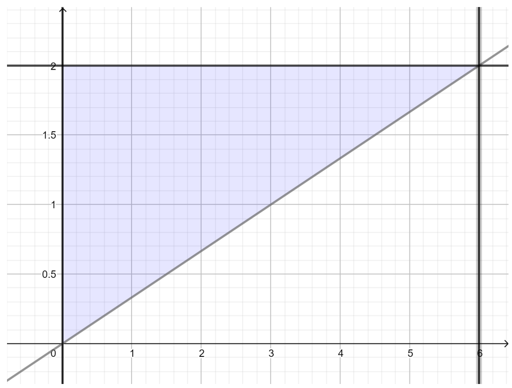
c)

$$\int_0^6 \int_{\frac{1}{3}x}^2 x^2\, \mathrm{e}^{y^4}\ \mathrm{d}y\ \mathrm{d}x = \int_0^2 \int_0^{3y} x^2\, \mathrm{e}^{y^4}\ \mathrm{d}x,\ \mathrm{d}y$$

$$= \int_0^2 \Big/_{0}^{3y} \frac{1}{3}x^3\, \mathrm{e}^{y^4}\ \mathrm{d}y$$

$$= \int_0^2 9y^3\, \mathrm{e}^{y^4}\ \mathrm{d}y = \Big/_{0}^{2} \frac{9}{4}\, \mathrm{e}^{y^4}$$

$$= \frac{9}{4}(\,\mathrm{e}^{16} - 1)$$



4.

$$\int_1^9 \int_1^{36x^2} \int_1^{\mathrm{e}^{x^2y^2}} \frac{3}{x^2y^2\sqrt{xy}} \cdot \frac{1}{z}\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x = \int_1^9 \int_1^{36x^2} \Big/_{1}^{\mathrm{e}^{x^2y^2}} \left(\frac{3}{x^2y^2\sqrt{xy}} \ln|z| \right)\ \mathrm{d}y\ \mathrm{d}x$$

$$= \int_1^9 \int_1^{36x^2} \frac{3}{x^2y^2\sqrt{xy}}(x^2y^2 - 0)\ \mathrm{d}y\ \mathrm{d}x$$

$$= \int_1^9 \left(\Big/_{1}^{36x^2}\ 6x^{-\frac{1}{2}}y^{\frac{1}{2}} \right)\ \mathrm{d}x$$

$$= \int_1^9 \left(6x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\ \mathrm{d}x = \Big/_{1}^{9} \left(4x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right)$$

$$= 6(4(9\sqrt{9} - 1) - 2(3 - 1)) = 6(4 \cdot 26 - 4) = 6 \cdot 4 \cdot 25 = 600.$$