MATEMATIIKAN PERUSKURSSI II

Ratkaisuja preppaustehtäviin

1. a) Muodostetaan osasummien jono

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\left[\frac{1}{(k+3)^2}-\frac{1}{(k+4)^2}\right] \\
&= \left[\frac{1}{4^2}-\frac{1}{5^2}\right]+\left[\frac{1}{5^2}-\frac{1}{6^2}\right]+\cdots+\left[\frac{1}{(n+2)^2}-\frac{1}{(n+3)^2}\right]+\left[\frac{1}{(n+3)^2}-\frac{1}{(n+4)^2}\right] \\
&= \frac{1}{16}-\frac{1}{(n+4)^2},\ n=1,2,\ldots,
\end{aligned}$$

josta saadaan

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{16}-\frac{1}{(n+4)^2}\right]=\frac{1}{16}.$$

Sarja siis suppenee ja sarjan summa on $\frac{1}{16}$.

b) Muodostetaan osasummien jono

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}(\sqrt[3]{k}-\sqrt[3]{k-1}) \\
&= (\sqrt[3]{1}-\sqrt[3]{0})+(\sqrt[3]{2}-\sqrt[3]{1})+(\sqrt[3]{3}-\sqrt[3]{2})+\cdots+(\sqrt[3]{n-1}-\sqrt[3]{n-2})+(\sqrt[3]{n}-\sqrt[3]{n-1}) \\
&= \sqrt[3]{n},\ n=1,2,\ldots,
\end{aligned}$$

josta saadaan

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\sqrt[3]{n}=\infty.$$

Sarja siis hajaantuu.

c) Vertailuperiaate:

$$\frac{2\sqrt{k}}{2k-1} > \frac{2\sqrt{k}}{2k} = \frac{1}{k^{\frac{1}{2}}} > 0$$

kaikilla $k = 1,\ 2,\ 3,\ \ldots$. Vertailusarja $\sum\limits_{k=1}^{\infty}\frac{1}{k^{\frac{1}{2}}}$ hajaantuu ($0 < p = \frac{1}{2} \leq 1$), joten tutkittava sarja hajaantuu vertailuperiaatteen nojalla.

d) Nyt $f(x) = x\,\mathrm{e}^{-x}$, $x \geq 1$. Integraalitestin oletukset ovat voimassa, sillä

- $f(x) = x\,\mathrm{e}^{-x} > 0$ kaikilla $x \geq 1$,
- f on jatkuva,
- $f'(x) = \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} = \mathrm{e}^{-x}(1-x) \leq 0$ kaikilla $x \geq 1$ (ja derivaatan ainoa nollakohta on kun $x = 1$), joten f on aidosti vähenevä.

Epäoleellinen integraali

$$\begin{aligned}
\lim_{M\to\infty}\int_1^M x\,\mathrm{e}^{-x}\,\mathrm{d}x &= \lim_{M\to\infty}\left(\Big/_{\!1}^{M} x(-\mathrm{e}^{-x}) + \int_1^M \mathrm{e}^{-x}\,\mathrm{d}x\right) \\
= \lim_{M\to\infty}(-x-1)\,\mathrm{e}^{-x} &= \lim_{M\to\infty}\big((-M-1)\,\mathrm{e}^{-M}-(-1-1)\,\mathrm{e}^{-1}\big) = \frac{2}{\mathrm{e}} < \infty
\end{aligned}$$

suppenee, joten integraalitestin nojalla tutkittava sarja suppenee.

e) Sarjan kaikki termit ovat negatiivisia.

f) Integraalitesti:
Tarkastellaan funktiota $f(x) = 2(2x-1)^{-1}$, kun $x \geq 1$.
Funktio $f(x) = 2(2x-1)^{-1} > 0$, kun $x \geq 1$.
Funktio $f(x)$ on vähenevä, sillä $f'(x) = -2(2x-1)^{-2}2 = -4(2x-1)^{-2} < 0$, kun $x \geq 1$.
Epäoleellinen integraali

$$\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty \frac{2}{2x-1}\,\mathrm{d}x = \Big/_1^\infty \ln|2x-1| = \infty.$$

Sarja hajaantuu integraalitestin nojalla.

Vertailuperiaate:

$$\frac{2}{2k-1} > \frac{2}{2k} = \frac{1}{k}$$

kaikilla $k = 1, 2, 3, \ldots$. Vertailusarja $\sum_{k=1}^{\infty} \frac{1}{k}$ hajaantuu $(0 < p = 1 \leq 1)$, joten tutkittava sarja $\sum_{k=1}^{\infty} \frac{2}{2k-1}$ hajaantuu vertailuperiaatteen nojalla.

g) Funktio $f(x) = 5(x+5)^{-2} > 0$, kun $x \geq 1$.

$$f'(x) = (-10)(x+5)^{-3} < 0,$$

kun $x \geq 1$, joten f on vähenevä funktio.

$$\int_1^\infty 5(x+5)^{-2}\,\mathrm{d}x = \Big/_1^\infty (-5)(x+5)^{-1} = 0 + \frac{5}{6} = \frac{5}{6} < \infty,$$

joten epäoleellinen integraali suppenee. Sarja suppenee integraalitestin nojalla.

Koska

$$0 < \frac{5}{(k+5)^2} < \frac{5}{k^2}$$

kaikilla $k = 1, 2, 3, \ldots$ ja vertailusarja $\sum_{k=1}^{\infty} \frac{5}{k^2}$ suppenee $(p = 2 > 1)$, tutkittava sarja suppenee vertailuperiaatteen nojalla.

2. a) Koska $R = \frac{1}{5}$, tutkittava potenssisarja suppenee varmasti, kun

$$\left|x - \frac{1}{5}\right| < \frac{1}{5} \quad -\frac{1}{5} < x - \frac{1}{5} < \frac{1}{5} \quad 0 < x < \frac{2}{5}.$$

b) Lasketaan raja-arvo:

$$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\frac{(-1)^{k+1}}{4^{k+1}}}{\frac{(-1)^k}{4^k}}\right| = \lim_{k\to\infty} \frac{4^k}{4^{k+1}} = \lim_{k\to\infty} \frac{1}{4} = \frac{1}{4}.$$

Suppenemissäde on

$$R = \frac{1}{L} = 4.$$

Koska potenssisarjan suppenemissäde $R = 4$, potenssisarja suppenee varmasti, kun $|x| < 4$. Piste $x = -2$ kuuluu tähän väliin, joten potenssisarja suppenee pisteessä $x = -2$.

c) Lasketaan raja-arvo:

$$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\frac{(-1)^{k+1}(k+2)}{2^{k+1}}}{\frac{(-1)^k(k+1)}{2^k}}\right| = \lim_{k\to\infty} \frac{k+2}{k+1} \cdot \frac{2^k}{2^{k+1}} = \lim_{k\to\infty} \frac{1+\frac{2}{k}}{1+\frac{1}{k}} \cdot \frac{1}{2} = \frac{1}{2}.$$

Suppenemissäde on $R = \frac{1}{L} = 2$.

3. a)

i) Tunnetaan potenssisarjat

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

joten saadaan

$$\lim_{x\to 0} \frac{x\cos(x) - \sin(x)}{\ln(1+x^3)}$$
$$= \lim_{x\to 0} \frac{x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) - (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)}{x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \cdots}$$
$$= \lim_{x\to 0} \frac{x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots}{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots}$$
$$= \lim_{x\to 0} \frac{x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots}{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots}$$
$$= \lim_{x\to 0} \frac{[(-\frac{1}{2!} + \frac{1}{3!})x^3 + (\frac{1}{4!} - \frac{1}{5!})x^5 + (-\frac{1}{6!} + \frac{1}{7!})x^7 + \cdots] : x^3}{(x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots) : x^3}$$
$$= \lim_{x\to 0} \frac{-\frac{1}{2!} + \frac{1}{3!} + (\frac{1}{4!} - \frac{1}{5!})x^2 + (-\frac{1}{6!} + \frac{1}{7!})x^4 + \cdots}{1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \cdots} = \frac{-\frac{1}{2} + \frac{1}{6}}{1} = -\frac{1}{3}.$$

ii) Tunnetaan Maclaurinin kehitelmät

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

joten saadaan

$$\begin{aligned}
&\lim_{x\to 0}\frac{x^2\ln(1+x)}{x\cos(3x)-\sin(x)}\\
&=\lim_{x\to 0}\frac{x^2\big(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\big)}{x\big(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\frac{(3x)^6}{6!}+\cdots\big)-\big(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\big)}\\
&=\lim_{x\to 0}\frac{x^3-\frac{x^4}{2}+\frac{x^5}{3}-\cdots}{x\big(1-\frac{3^2x^2}{2!}+\frac{3^4x^4}{4!}-\frac{3^6x^6}{6!}+\cdots\big)-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots}\\
&=\lim_{x\to 0}\frac{x^3-\frac{x^4}{2}+\frac{x^5}{3}-\cdots}{x-\frac{3^2x^3}{2!}+\frac{3^4x^5}{4!}-\frac{3^6x^7}{6!}+\cdots-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots}\\
&=\lim_{x\to 0}\frac{\big(x^3-\frac{x^4}{2}+\frac{x^5}{3}-\cdots\big):x^3}{\big((\frac{1}{3!}-\frac{3^2}{2!})x^3+(\frac{3^4}{4!}-\frac{1}{5!})x^5+(\frac{1}{7!}-\frac{3^6}{6!})x^7+\cdots\big):x^3}\\
&=\lim_{x\to 0}\frac{1-\frac{x}{2}+\frac{x^2}{3}-\cdots}{\frac{1}{3!}-\frac{3^2}{2!}+(\frac{3^4}{4!}-\frac{1}{5!})x^2+(\frac{1}{7!}-\frac{3^6}{6!})x^4+\cdots}=\frac{1}{\frac{1}{6}-\frac{9}{2}}=-\frac{3}{13}
\end{aligned}$$

b) Nyt $n=1$ ja edelleen $f(x)=(x+1)^{\frac{1}{3}}$, $f(0)=1$, $f'(x)=\frac{1}{3}(x+1)^{-\frac{2}{3}}$ ja $f'(0)=\frac{1}{3}$. Sijoitetaan nämä suoraan annettuun kaavaan, jolloin

$$T_1(x)=\sum_{k=0}^{1}\frac{f^{(k)}(0)}{k!}x^k=\frac{f(0)}{0!}+\frac{f'(0)}{1!}=1+\frac{1}{3}x.$$

Funktion f arvon arviointi kun $x=0.331$:

$$f(0.331)\approx T_1(0.331)=1+\frac{1}{3}\cdot 0.331=1.110333.$$

4. b) Valitaan ensin lähestymistie $y=x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}}\frac{6x^2y^4}{x^3+3y^{12}}=\lim_{x\to 0}\frac{6x^2x^4}{x^3+3x^{12}}=\lim_{x\to 0}\frac{6x^6}{x^3(1+3x^9)}=\lim_{x\to 0}\frac{6x^3}{1+3x^9}=\frac{0}{1+0}=0.$$

Valitaan sitten lähestymistie $y=x^{\frac{1}{4}}$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}}\frac{x^2y}{x^4+y^2}=\lim_{x\to 0}\frac{6x^2(x^{\frac{1}{4}})^4}{x^3+3(x^{\frac{1}{4}})^{12}}=\lim_{x\to 0}\frac{6x^3}{x^3+3x^3}=\lim_{x\to 0}\frac{6}{4}=\frac{3}{2}.$$

Koska eri lähestymisteitä pitkin saatiin eri arvot, niin raja-arvoa ei ole olemassa origossa.

5. a) Koska

$$\vec{u}^0=\frac{\vec{u}}{|\vec{u}|}=\frac{(1,-2)}{\sqrt{1^2+(-2)^2}}=(\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}})$$

ja $f_x(x,y) = -4x,\ f_x(x,y) = 3y^2$, niin

$$\nabla_{\vec{u}^0} f(-1,2) = \nabla f(-1,2) \cdot \vec{u}^0 = (4,12) \cdot (\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) = \frac{4-24}{\sqrt{5}} = -4\sqrt{5}.$$

b) Koska $f_x(x,y) = 5(x-2)^4 y$ ja $f_y(x,y) = (x-2)^5$, niin $f_x(3,-1) = -5$ ja $f_y(3,-1) = 1$. Edelleen $\vec{u} = -4\vec{i} + 3\vec{j}$, joten $|\vec{u}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$, ja siis $\vec{u}^0 = \frac{\vec{u}}{|\vec{u}|} = -\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$. Tällöin

$$\begin{aligned}\nabla_{\vec{u}^0} f(3,-1) &= \nabla f(3,-1) \cdot \vec{u}^0 = (f_x(3,-1), f_y(3,-1)) \cdot \vec{u}^0 \\ &= (-5,1) \cdot (-\frac{4}{5}, \frac{3}{5}) = \frac{23}{5} = 4\frac{3}{5}.\end{aligned}$$

c) Funktion $f(x,y) = x\ln(y^2+1)$ osittaisderivaatat ovat $f_x(x,y) = \ln(y^2+1)$ ja $f_y(x,y) = \frac{2xy}{y^2+1}$, joten

$$\nabla f(1,-1) = (f_x(1,-1), f_y(1,-1)) = (\ln(2), -1).$$

Koska $\vec{u} = 4\vec{i} - 3\vec{j}$, niin

$$\vec{u}^0 = \frac{\vec{u}}{|\vec{u}|} = \frac{(4,-3)}{\sqrt{4^2 + (-3)^2}} = \frac{(4,-3)}{5}.$$

Siis

$$\nabla_{\vec{u}^0} f(1,-1) = \nabla f(1,-1) \cdot \vec{u}^0 = (\ln(2), -1) \cdot \frac{(4,-3)}{5} = \frac{\ln(16)+3}{5}.$$

d) Funktion $f(x,y) = 4x^2y - xy^3 - 5$ gradientti on

$$\nabla f(x,y) = \begin{bmatrix} 8xy - y^3 \\ 4x^2 - 3xy^2 \end{bmatrix}.$$

Normeerataan vektori $\vec{u}$:

$$\vec{u}^0 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Gradientti pisteessä $(-1,2)$:

$$\nabla f(-1,2) = \begin{bmatrix} -8 \cdot 2 - 8 \\ 4 + 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} -24 \\ 16 \end{bmatrix}$$

Suunnattu derivaatta pisteessä $(-1,2)$ on vektorien $\nabla f(-1,2)$ ja $\vec{u}^0$ pistetulo:

$$\nabla_{\vec{u}^0} f(-1,2) = -24 \cdot (-\frac{1}{\sqrt{2}}) + 16 \cdot (-\frac{1}{\sqrt{2}}) = \frac{8}{\sqrt{2}} = 4\sqrt{2}.$$

Funktion arvot vähenevät voimakkaimmin pisteessä $(-1,2)$ negatiivisen gradientin suuntaan eli suuntaan $-\nabla f(-1,2) = 8\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

e) Koska funktion arvot eivät muutu tasa-arvokäyrää pitkin kuljettaessa, niin funktion arvot muuttuvat vähiten vektorin $\vec{v}$ suuntaan. Täten funktion arvot vähenevät voimakkaimmin vektorin $\vec{u}$ suuntaan.

6. a) Koska $f_x(x,y) = \frac{3}{x+1}$, $f_y(x,y) = -\mathrm{e}^{y-2}$ ja $z(t) = f(x(t), y(t))$, niin

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3}{x+1}\ 1 - \mathrm{e}^{y-2}\ 2t = \frac{3}{t-1+1}\ 1 - \mathrm{e}^{t^2+2-2}\ 2t = 3t^{-1} - 2t\,\mathrm{e}^{t^2}.$$

b) Osittaisderivaatat ovat $f_x(x,y) = y^{-1}$ ja $f_y(x,y) = -xy^{-2}$. Lisäksi $z(s,t) = f(x(s,t), y(s,t))$ ja

$$x_t = \mathrm{e}^s, y_t = \mathrm{e}^{-s},$$

joten ketjusäännön avulla saadaan

$$\begin{aligned}\frac{\partial z}{\partial t} &= z_t = f_x x_t + f_y y_t = y^{-1}\,\mathrm{e}^s - xy^{-2}\,\mathrm{e}^{-s} = (t\,\mathrm{e}^{-s})^{-1}\,\mathrm{e}^s - t\,\mathrm{e}^s(t\,\mathrm{e}^{-s})^{-2}\,\mathrm{e}^{-s} \\ &= t^{-1}\,\mathrm{e}^s\,\mathrm{e}^s - t\,\mathrm{e}^s t^{-2}\,\mathrm{e}^{2s}\,\mathrm{e}^{-s} = t^{-1}\,\mathrm{e}^{2s} - t^{-1} + \mathrm{e}^{2s} = 0.\end{aligned}$$

c) Osittaisderivaatat ovat $f_x(x,y) = 1\,\mathrm{e}^x + x\,\mathrm{e}^x = (x+1)\,\mathrm{e}^x$ ja $f_y(x,y) = -y^6$. Lisäksi $z(t) = f(x(t), y(t))$ ja

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}[\ln(t)]}{\mathrm{d}t} = t^{-1}, \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}(t^{\frac{1}{3}})}{\mathrm{d}t} = \frac{1}{3}t^{-\frac{2}{3}},$$

joten ketjusäännön avulla

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = (x+1)\,\mathrm{e}^x t^{-1} - y^6\frac{1}{3}t^{-\frac{2}{3}} \\ &= (\ln(t)+1)\,\mathrm{e}^{\ln(t)}t^{-1} - (\sqrt[3]{t})^6\frac{1}{3}t^{-\frac{2}{3}} = [\ln(t)+1]tt^{-1} - t^2\frac{1}{3}t^{-\frac{2}{3}} = \ln(t) + 1 - \frac{t\sqrt[3]{t}}{3}.\end{aligned}$$

7. a) Koska

$$\begin{aligned}f_x(x,y) &= 12y - 3y^2 - 3x^2, \\ f_y(x,y) &= 12x - 6xy,\end{aligned}$$

saadaan toisiksi osittaisderivaatoiksi

$$\begin{aligned}f_{xx}(x,y) &= -6x, \\ f_{yy}(x,y) &= -6x, \\ f_{xy}(x,y) &= 12 - 6y.\end{aligned}$$

Tällöin $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 36x^2 - (12-6y)^2$.
Piste $(0,0)$ on satulapiste, sillä $D(0,0) = -144 < 0$.
Piste $(0,4)$ on satulapiste, sillä $D(0,4) = -144 < 0$.
Piste $(-2,2)$ ei ole satulapiste, sillä $D(-2,2) = 144 > 0$.
Piste $(2,2)$ ei ole satulapiste, sillä $D(2,2) = 144 > 0$.

b) Koska $f_x(x,y) = 6x^2 + 6x - 12$ ja $f_y(x,y) = 2y$, niin

$$\begin{aligned}f_{xx}(x,y) &= 12x + 6, \\ f_{yy}(x,y) &= 2, \\ f_{xy}(x,y) &= 0.\end{aligned}$$

Nyt $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 12(2x+1)$.
Piste $(-2,0)$ on satulapiste, sillä $D(-2,0) = -36 < 0$. Piste $(1,0)$ on paikallinen minimipiste, sillä $D(1,0) = 36 > 0$, $f_{xx}(1,0) = 18 > 0$.

c) Koska $\nabla f(x,y) = (f_x(x,y), f_y(x,y)) = (12y - 6xy, 12x - 3x^2 - 3y^2)$, niin

$$\begin{aligned}\nabla f(4,0) &= (12\cdot 0 - 6\cdot 4\cdot 0, 12\cdot 4 - 3\cdot 4^2 - 3\cdot 0^2) = (0,0) = \vec{0}, \\ \nabla f(2,-2) &= (12\cdot(-2) - 6\cdot 2\cdot(-2), 12\cdot 2 - 3\cdot 2^2 - 3\cdot(-2)^2) = (0,0) = \vec{0}.\end{aligned}$$

Lasketaan toiset osittaisderivaatat

$$\begin{aligned}f_{xx}(x,y) &= -6y, \\ f_{yy}(x,y) &= -6y, \\ f_{xy}(x,y) &= 12 - 6x.\end{aligned}$$

Nyt $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = (-6y)(-6y) - (12-6x)^2 = -36x^2 + 36y^2 + 144x - 144$.
Piste $(4,0)$ on satulapiste, sillä $D(4,0) = -144 < 0$.
Piste $(2,-2)$ on paikallinen minimipiste, sillä

$$\begin{aligned} D(2,-2) &= 144 > 0, \\ f_{xx}(2,-2) &= 12 > 0. \end{aligned}$$

d) Kriittiset pisteet ovat gradientin nollakohtia:

$$\begin{cases} f_x(x,y) &= 4x - 4xy &= 0, \\ f_y(x,y) &= -2x^2 + 2y &= 0. \end{cases}$$

Tulon nollasäännöllä saadaan 1. yhtälöstä $x = 0$ tai $y = 1$.
Sijoitetaan 2. yhtälöön $x = 0$: $2y = 0 \;\Rightarrow\; y = 0$.
Sijoitetaan 2. yhtälöön $y = 1$: $-2x^2 + 2 = 0 \;\Rightarrow\; x = \pm 1$.
Kriittiset pisteet ovat $(x,y) = (0,0)$, $(x,y) = (-1,1)$ ja $(x,y) = (1,1)$.
Lasketaan toiset osittaisderivaatat

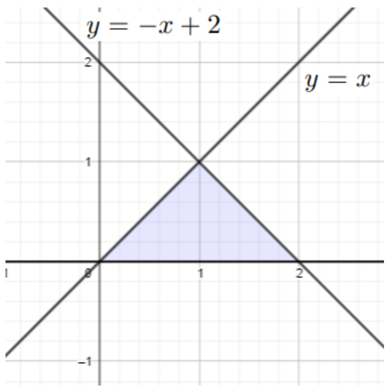
$$\begin{aligned} f_{xx}(x,y) &= 4 - 4y, \\ f_{yy}(x,y) &= 2, \\ f_{xy}(x,y) &= -4x. \end{aligned}$$

Nyt $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 8(1-y) - 16x^2$.
Piste $(0,0)$ on paikallinen minimipiste, sillä $D(0,0) = 8 > 0$ ja $f_{xx}(0,0) = 4 > 0$.
Piste $(-1,1)$ on satulapiste, sillä $D(-1,1) = -16 < 0$.
Piste $(1,1)$ on myös satulapiste, sillä $D(1,1) = -16 < 0$.

8. a)

$$\iint_A 6xy \,\mathrm{d}A = \int_0^4 \int_{\sqrt{x}}^2 6xy \;\mathrm{d}y\;\mathrm{d}x = \int_0^4 \Big/_{\sqrt{x}}^{2} 3xy^2 \;\mathrm{d}x$$
$$= \int_0^4 (12x - 3x^2)\;\mathrm{d}x = \Big/_0^4 (6x^2 - x^3) = 32$$

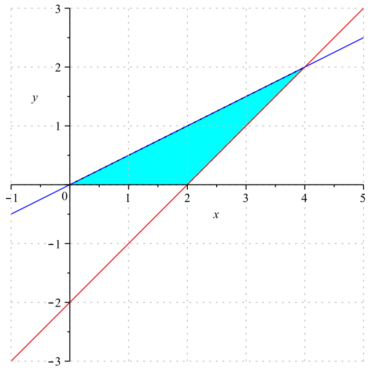
b) $A = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1, y \le x \le 2-y\}$



$$\iint_A \frac{2}{-y^2+2y+4}\,\mathrm{d}A = \int_0^1 \int_y^{2-y} \frac{2}{-y^2+2y+4}\;\mathrm{d}x\;\mathrm{d}y$$
$$= \int_0^1 \Big/_y^{2-y} \frac{2x}{-y^2+2y+4}\;\mathrm{d}y = \int_0^1 \frac{2(2-y-y)}{-y^2+2y+4}\;\mathrm{d}y$$
$$= 2\int_0^1 \frac{2-2y}{-y^2+2y+4}\;\mathrm{d}y$$
$$= 2 \Big/_0^1 \ln\left|-y^2+2y+4\right| = 2\ln(\frac{5}{4})$$

c)

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 2, 2y \leq x \leq y+2\}$$



$$\iint_A 12x(-y^3+2y^2+4y)^5 \,\mathrm{d}A = \int_0^2 \int_{2y}^{y+2} 12x(-y^3+2y^2+4y)^5 \,\mathrm{d}x \,\mathrm{d}y$$

$$= \int_0^2 \Big/_{2y}^{y+2} 6x^2(-y^3+2y^2+4y)^5 \,\mathrm{d}y = \int_0^2 6((y+2)^2-(2y)^2)(-y^3+2y^2+4y)^5 \,\mathrm{d}y$$

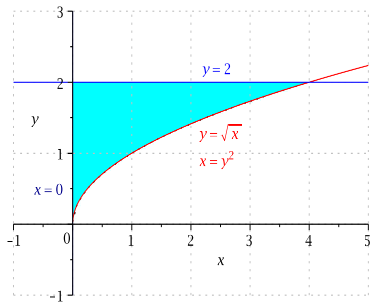
$$= \int_0^2 6(-3y^2+4y+4)(-y^3+2y^2+4y)^5 \,\mathrm{d}y = \Big/_0^2 (-y^3+2y^2+4y)^6 = 8^6 = 262144$$

9. a)

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{y^3+1} \,\mathrm{d}y \,\mathrm{d}x = \int_0^2 \int_0^{y^2} \frac{3}{y^3+1} \,\mathrm{d}x \,\mathrm{d}y$$

$$= \int_0^2 \left(\Big/_0^{y^2} \frac{3x}{y^3+1} \right) \mathrm{d}y = \int_0^2 \frac{3y^2}{y^3+1} \,\mathrm{d}y$$

$$= \Big/_0^2 \ln\left|y^3+1\right| = \ln(9)$$



b) Integroimisalue A voidaan kuvata kahdella eri tavalla:

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, 3x \leq y \leq 6\} \text{ tai } A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 6, 0 \leq x \leq \tfrac{1}{3}y\}.$$

Tällöin voidaan vaihtaa integrointijärjestys ja laskea

$$\int_0^2 \int_{3x}^6 54x(y^3+1)^{-2} \,\mathrm{d}y \,\mathrm{d}x = \int_0^6 \int_0^{\frac{1}{3}y} 54x(y^3+1)^{-2} \,\mathrm{d}x \,\mathrm{d}y = \int_0^6 \Big/_0^{\frac{1}{3}y} 27x^2(y^3+1)^{-2} \,\mathrm{d}y$$

$$= \int_0^6 3y^2(y^3+1)^{-2} \,\mathrm{d}y = \Big/_0^6 \left(-\frac{1}{y^3+1}\right) = \frac{216}{217}.$$

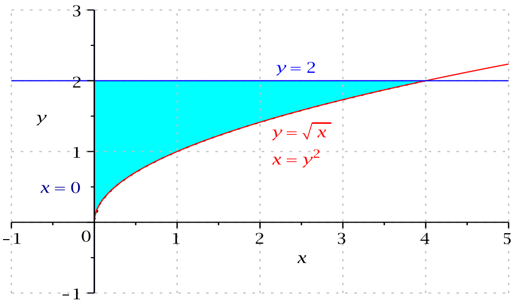
c)

$$\int_0^4 \int_{\sqrt{x}}^2 84x^2(y^7+1)^3 \; \mathrm{d}y \; \mathrm{d}x = \int_0^2 \int_0^{y^2} 84x^2(y^7+1)^3 \; \mathrm{d}x \; \mathrm{d}y$$

$$= \int_0^2 \left(\Big/_{0}^{y^2} 28x^3(y^7+1)^3 \right) \; \mathrm{d}y = \int_0^2 28(y^2)^3(y^7+1)^3 \; \mathrm{d}y$$

$$= \int_0^2 28y^6(y^7+1)^3 \; \mathrm{d}y = \Big/_{0}^{2} (y^7+1)^4 = 129^4 - 1$$

$$= 276922880$$



d)

$$\int_{-4}^0 \int_{-\frac{x}{4}}^1 3x^2 \, \mathrm{e}^{-y^4} \; \mathrm{d}y \; \mathrm{d}x = \int_0^1 \int_{-4y}^0 3x^2 \, \mathrm{e}^{-y^4} \; \mathrm{d}x \; \mathrm{d}y$$

$$= \int_0^1 \Big/_{-4y}^{0} x^3 \, \mathrm{e}^{-y^4} \; \mathrm{d}y = \int_0^1 64y^3 \, \mathrm{e}^{-y^4} \; \mathrm{d}y$$

$$= (-16) \int_0^1 (-4y^3) \, \mathrm{e}^{-y^4} \; \mathrm{d}y = (-16) \Big/_{0}^{1} \; \mathrm{e}^{-y^4}$$

$$= 16(1 - \mathrm{e}^{-1})$$

10. a) Napakoordinaateissa $x = r\cos\varphi$ ja $y = r\sin\varphi$, joten

$$\iint_A \frac{2x}{x^2+y^2} \, \mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_1^4 \frac{2r\cos\varphi}{r^2} \, r \; \mathrm{d}r \; \mathrm{d}\varphi = \int_0^{\frac{\pi}{2}} \int_1^4 2\cos\varphi \; \mathrm{d}r \; \mathrm{d}\varphi$$

$$= \int_0^{\frac{\pi}{2}} \Big/_{1}^{4} 2\cos\varphi r \; \mathrm{d}\varphi = \int_0^{\frac{\pi}{2}} 6\cos\varphi \, \mathrm{d}\varphi = \Big/_{0}^{\frac{\pi}{2}} 6\sin\varphi = 6.$$

b) Napakoordinaateissa $x = r\cos\varphi$ ja $y = r\sin\varphi$, joten

$$\iint_A \frac{2x-3y}{x^2+y^2} \, \mathrm{d}A = \int_\pi^{2\pi} \int_1^2 \frac{2r\cos\varphi - 3r\sin\varphi}{r^2} \, r \; \mathrm{d}r \; \mathrm{d}\varphi = \int_\pi^{2\pi} \int_1^2 (2\cos\varphi - 3\sin\varphi) \; \mathrm{d}r \; \mathrm{d}\varphi$$

$$= \int_\pi^{2\pi} \Big/_{1}^{2} (2\cos\varphi - 3\sin\varphi) r \; \mathrm{d}\varphi = \int_\pi^{2\pi} (2\cos\varphi - 3\sin\varphi)(2-1) \; \mathrm{d}\varphi = \Big/_{\pi}^{2\pi} (2\sin\varphi + 3\cos\varphi)$$

$$= 0 + 3 - 0 + 3 = 6.$$

c) Alue $A = \{(x,y) \in \mathbb{R}^2 : -3 \leq x \leq 0, 0 \leq y \leq \sqrt{9-x^2}\}$ on napakoordinaateissa $A' = \{(r,\varphi) \in \mathbb{R}^2 : 0 \leq r \leq 3, \frac{\pi}{2} \leq \varphi \leq \pi\}$. Tällöin

$$\int_{-3}^{0} \int_{0}^{\sqrt{9-x^2}} \frac{4}{4+x^2+y^2}\, \mathrm{d}y\, \mathrm{d}x = \iint_A \frac{4}{4+x^2+y^2}\, \mathrm{d}A = \iint_{A'} \frac{4}{4+r^2} r\, \mathrm{d}r\, \mathrm{d}\varphi$$
$$= \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{3} \frac{4}{4+r^2} r\, \mathrm{d}r\, \mathrm{d}\varphi = \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{3} \frac{4r}{4+r^2}\, \mathrm{d}r\, \mathrm{d}\varphi$$
$$= \int_{\frac{\pi}{2}}^{\pi} \bigg/_{0}^{3} 2\ln(4+r^2)\, \mathrm{d}\varphi = \int_{\frac{\pi}{2}}^{\pi} 2\ln\frac{13}{4}\, \mathrm{d}\varphi = \pi \ln\frac{13}{4}.$$

d) Integroimisalue $A = \{(x,y) \in \mathbb{R}^2 : -4 \leq x \leq 0, -\sqrt{16-x^2} \leq y \leq 0\}$ on napakoordinaateissa alue $A' = \{(r,\varphi) \in \mathbb{R}^2 : 0 \leq r \leq 4, \pi \leq \varphi \leq \frac{3\pi}{2}\}$.

$$\int_{-4}^{0} \int_{-\sqrt{16-x^2}}^{0} \frac{2}{x^2+y^2+2}\, \mathrm{d}y\, \mathrm{d}x = \iint_A \frac{2}{x^2+y^2+2}\, \mathrm{d}A = \iint_{A'} \frac{2}{r^2+2} r\, \mathrm{d}r\, \mathrm{d}\varphi$$
$$= \int_{\pi}^{\frac{3\pi}{2}} \int_{0}^{4} \frac{2r}{r^2+2}\, \mathrm{d}r\, \mathrm{d}\varphi = \int_{\pi}^{\frac{3\pi}{2}} \bigg/_{0}^{4} \ln(r^2+2)\, \mathrm{d}\varphi$$
$$= \int_{\pi}^{\frac{3\pi}{2}} (\ln(18) - \ln(2))\, \mathrm{d}\varphi = \bigg/_{\pi}^{\frac{3\pi}{2}} \ln(9)\varphi = \frac{\pi}{2}\ln(9)$$

e) $A = A_1 \cup A_2$, missä $A_1 = \{(x,y) \in \mathbb{R}^2 : -3 \leq x \leq 0, 0 \leq y \leq 2\}$ ja $A_2 = \{(r,\varphi) \in \mathbb{R}^2 : 0 \leq r \leq 2, 0 \leq \varphi \leq \frac{\pi}{2}\}$.

$$\iint_A 3(x^2+y^2)\, \mathrm{d}A = \iint_{A_1} 3(x^2+y^2)\, \mathrm{d}A + \iint_{A_2} 3(x^2+y^2)\, \mathrm{d}A$$
$$= \int_{-3}^{0} \int_{0}^{2} 3(x^2+y^2)\, \mathrm{d}y\, \mathrm{d}x + \int_{0}^{\frac{\pi}{2}} \int_{0}^{2} 3r^2 r\, \mathrm{d}r\, \mathrm{d}\varphi = \int_{-3}^{0} \bigg/_{0}^{2} (3x^2y + y^3)\, \mathrm{d}x + \int_{0}^{\frac{\pi}{2}} \bigg/_{0}^{2} \frac{3}{4} r^4\, \mathrm{d}\varphi$$
$$= \int_{-3}^{0} (6x^2+8)\, \mathrm{d}x + \int_{0}^{\frac{\pi}{2}} 12\, \mathrm{d}\varphi = \bigg/_{-3}^{0} (2x^3+8x) + \bigg/_{0}^{\frac{\pi}{2}} 12\varphi = 78 + 6\pi$$

11. a) Lasketaan kappaleen tilavuus:

$$\iiint_V 1\, \mathrm{d}V = \int_{0}^{2\pi} \int_{0}^{5} \int_{r-5}^{25-r^2} r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\varphi = \int_{0}^{2\pi} \int_{0}^{5} \Big(\bigg/_{r-5}^{25-r^2} rz \Big)\, \mathrm{d}r\, \mathrm{d}\varphi$$
$$= \int_{0}^{2\pi} \int_{0}^{5} [r(25-r^2) - r(r-5)]\, \mathrm{d}r\, \mathrm{d}\varphi = \int_{0}^{2\pi} \int_{0}^{5} (-r^3 - r^2 + 30r)\, \mathrm{d}r\, \mathrm{d}\varphi$$
$$= \int_{0}^{2\pi} \bigg/_{0}^{5} (-\frac{1}{4}r^4 - \frac{1}{3}r^3 + 15r^2)\, \mathrm{d}\varphi = \int_{0}^{2\pi} (-\frac{625}{4} - \frac{125}{3} + 375)\, \mathrm{d}\varphi = \frac{2125\pi}{6}.$$

b) Annettu kappale sylinterikoordinaateissa on

$$V' = \{(r, \varphi, z) \in \mathbb{R}^3 : 0 \le r \le 3, 0 \le \varphi \le \pi, 0 \le z \le 3 - r\}.$$

$$\iiint_V 20z\sqrt{x^2+y^2}\,\mathrm{d}V = \iiint_{V'} 20zr\,r\,\mathrm{d}V = \int_0^{\pi}\int_0^{3}\int_0^{3-r} 20zr\,r\ \mathrm{d}z\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{\pi}\int_0^{3} \Big/_0^{3-r} 10z^2r^2\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{\pi}\int_0^{3} 10(3-r)^2r^2\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{\pi}\int_0^{3} 10(9-6r+r^2)r^2\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{\pi}\int_0^{3}(90r^2-60r^3+10r^4)\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{\pi} \Big/_0^{3}(30r^3-15r^4+2r^5)\,\mathrm{d}\varphi = 27(30-45+18)\pi = 81\pi$$

c) Kappale voidaan kuvata sylinterikoordinaattien avulla seuraavasti:

$$V' = \{(r, \varphi, z) \in \mathbb{R}^3 : 0 \le r \le 3, 0 \le \varphi \le 2\pi, 0 \le z \le 9 - r^2\}.$$

Lasketaan integroimalla tilavuus:

$$\iiint_V 1\,\mathrm{d}V = \iiint_{V'} r\ \mathrm{d}r\,\mathrm{d}\varphi\ \mathrm{d}z = \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r\ \mathrm{d}z\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^{3} \Big/_0^{9-r^2} rz\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^{3}(-r^3+9r)\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi} \Big/_0^{3}(-\frac{1}{4}r^4+\frac{9}{2}r^2)\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi}(-\frac{81}{4}+\frac{81}{2})\,\mathrm{d}\varphi = \frac{81\pi}{2}.$$

d) Kappale voidaan kuvata sylinterikoordinaattien avulla seuraavasti:

$$V' = \{(r, \varphi, z) \in \mathbb{R}^3 : 0 \le r \le 2, 0 \le \varphi \le 2\pi, r - 2 \le z \le 4 - r^2\}.$$

Lasketaan kappaleen tilavuus:

$$\iiint_V 1\,\mathrm{d}V = \iiint_{V'} r\ \mathrm{d}r\,\mathrm{d}\varphi\ \mathrm{d}z = \int_0^{2\pi}\int_0^{2}\int_{r-2}^{4-r^2} r\ \mathrm{d}z\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^{2}(\Big/_{r-2}^{4-r^2} rz)\ \mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^{2}[r(4-r^2)-r(r-2)]\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^{2}(-r^3-r^2+6r)\ \mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \Big/_0^{2}(-\frac{1}{4}r^4-\frac{1}{3}r^3+3r^2)\,\mathrm{d}\varphi = \int_0^{2\pi}(-4-\frac{8}{3}+12)\,\mathrm{d}\varphi$$

$$= \frac{32\pi}{3}.$$

12. a)

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ xy & y^2 - z^2 & yz \end{vmatrix} = (z + 2z)\vec{i} - (0 - 0)\vec{j} + (0 - x)\vec{k} = 3z\vec{i} - x\vec{k} = (3z, 0, -x)$$

$$\nabla f = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k} = 6x\vec{i} - 12y^2\vec{j} + 2\vec{k} = (6x, -12y^2, 2)$$

b)

$$\begin{cases} (3t-1, 4t, t-2) & = (-1, 0, -2), \quad \text{käyrän alkupiste} \\ (3t-1, 4t, t-2) & = (2, 4, -1), \quad \text{käyrän loppupiste} \end{cases} ; \begin{cases} t & = 0, \quad \text{käyrän alkupiste} \\ t & = 1. \quad \text{käyrän loppupiste} \end{cases}$$

Siis $0 \le t \le 1$ ja koska $\vec{x}'(t) = 3\vec{i} + 4\vec{j} + \vec{k}$, niin

$$\begin{aligned} \int_C 2xy \, \mathrm{d}x - y(z+2) \, \mathrm{d}y + 4y \, \mathrm{d}z &= \int_0^1 [2(3t-1)(4t)3 - 4t(t-2+2)4 + 4(4t)] \, \mathrm{d}t \\ &= \int_0^1 (72t^2 - 24t - 16t^2 + 16t) \, \mathrm{d}t = \int_0^1 (56t^2 - 8t) \, \mathrm{d}t \\ &= \Big/_0^1 (\frac{56}{3}t^3 - 4t^2) = \frac{56}{3} - 4 = \frac{44}{3} = 14\frac{2}{3}. \end{aligned}$$

Vektorikenttä $\vec{F}(x, y, z) = 2xy\vec{i} - (yz + 2y)\vec{j} + 4y\vec{k}$ ei ole konservatiivinen, sillä

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & -yz - 2y & 4y \end{vmatrix} = (4 - (-y))\vec{i} - (0 - 0)\vec{j} + (0 - 2x)\vec{k} = (4 + y)\vec{i} - 2x\vec{k} \neq \vec{0}.$$

Potentiaalifunktiota $U = U(x, y, z)$ ei ole olemassa, joten käyräintegraalin arvoa ei voi laskea potentiaalifunktion $U(x, y, z)$ avulla.

c) Lasketaan potentiaalifunktio:

$$\begin{aligned} U_x(x, y) &\overset{(1)}{=} 2x - 3x^2y, \\ U_y(x, y) &\overset{(2)}{=} -x^3 + 3y^2, \end{aligned}$$

joten

$$\begin{aligned} (1): \qquad U(x, y) &= \int (2x - 3x^2y) \, \mathrm{d}x = x^2 - x^3y + A(y) \\ U_y &= -x^3 + A'(y) \overset{(2)}{=} -x^3 + 3y^2 \\ & \quad A'(y) = 3y^2 \qquad A(y) = \int 3y^2 \, \mathrm{d}y = y^3 + C \\ U(x, y) &= x^2 - x^3y + y^3 + C. \end{aligned}$$

Koska $\vec{x}(0) = 0e^0\vec{i} + 2\sqrt{0}\vec{j} = (0, 0)$ (alkupiste) ja $\vec{x}(1) = 1\,\mathrm{e}^1\vec{i} + 2\sqrt{1}\vec{j} = (\mathrm{e}, 2)$ (loppupiste), niin

$$\begin{aligned} \int_C (2x - 3x^2y) \, \mathrm{d}x + (-x^3 + 3y^2) \, \mathrm{d}y &= U(\mathrm{e}, 2) - U(0, 0) \\ &= \mathrm{e}^2 - \mathrm{e}^3 2 + 2^3 + C - C = \mathrm{e}^2 - 2\,\mathrm{e}^3 + 8. \end{aligned}$$

d)

$$\begin{aligned}
U_x(x,y,z) &\overset{(1)}{=} x+2y+4z,\\
U_y(x,y,z) &\overset{(2)}{=} 2x-3y-z,\\
U_z(x,y,z) &\overset{(3)}{=} 4x-y+2z.
\end{aligned}$$

Lasketaan potentiaalifunktio:

$$\begin{array}{llll}
(1): & U(x,y,z) & = & \int (x+2y+4z)\ \mathrm{d}x \\
 & & = & \frac{1}{2}x^2+2xy+4xz+A(y,z) \\
 & U_y & = & 2x+\frac{\partial A}{\partial y}(y,z) \\
 & & \overset{(2)}{=} & 2x-3y-z \\
 & & & \frac{\partial A}{\partial y}(y,z) = -3y-z \qquad A(y,z) = \int(-3y-z)\ \mathrm{d}y \\
 & & & \qquad\qquad = -\frac{3}{2}y^2-yz+B(z) \\
 & U(x,y,z) & = & \frac{1}{2}x^2+2xy+4xz-\frac{3}{2}y^2-yz+B(z) \\
 & U_z & = & 4x-y+B'(z) \\
 & & \overset{(3)}{=} & 4x-y+2z. \\
 & & & B'(z)=2z \qquad B(z)=\int 2z\ \mathrm{d}z = z^2+C \\
 & U(x,y,z) & = & \frac{1}{2}x^2+2xy+4xz \\
 & & & -\frac{3}{2}y^2-yz+z^2+C.
\end{array}$$

Koska $\vec{x}(0) = (2\cdot 0, -0^2, 3\cdot 0^3) = (0,0,0)$ (loppupiste) ja $\vec{x}(-1) = (2(-1), -(-1)^2, 3(-1)^3) = (-2,-1,-3)$ (alkupiste), niin

$$\begin{aligned}
\int_C (x+2y+4z)\ \mathrm{d}x + (2x-3y-z)\ \mathrm{d}y + (4x-y+2z)\ \mathrm{d}z &= U(0,0,0) - U(-2,-1,-3)\\
&= 0+C-(2+4+24-\frac{3}{2}-3+9+C)\\
&= -34\frac{1}{2}.
\end{aligned}$$

e) Käyrän parametriesityksestä saadaan $t_0 = 0$ ja $t_1 = \pi$ ja $\vec{x}'(t) = -a\sin(t)\vec{i} + a\cos(t)\vec{j}$. Käyräintegraalin arvo

$$\begin{aligned}
\int_C (x-y)\,\mathrm{d}x + (y^3+x)\,\mathrm{d}y &= \int_0^\pi [(a\cos(t)-a\sin(t))(-a\sin(t)) + (a^3\sin^3(t)+a\cot(t))(a\cos(t))]\,\mathrm{d}t\\
&= \int_0^\pi [-a^2\sin(t)\cos(t) + a^2\sin^2(t) + a^3\sin^3(t)\cos(t) + a^2\cos^2(t)]\,\mathrm{d}t\\
&= \int_0^\pi [a^2 - a^2\sin(t)\cos(t) + a^3\sin^3(t)\cos(t)]\,\mathrm{d}t\\
&= \Big/_0^\pi\ a^2t - \frac{a^2}{2}\sin^2(t) + \frac{a^3}{4}\sin^4(t) = a^2\pi.
\end{aligned}$$

Vektorikentän divergenssi $\mathrm{div}\vec{F} = (\partial_x, \partial_y, \partial_z)\cdot(x-y, y^3+x, 0) = 1+3y^2$.

13. a)

$$A = \{(x,y)\in\mathbb{R}^2 : -1 \le x \le 1, 3x^2 \le y \le 3\},$$

joten Greenin lauseen nojalla

$$\oint_{\partial A}(-4xy)\ \mathrm{d}x+(5x-y^2)\ \mathrm{d}y=\iint_A \big(\frac{\partial}{\partial x}(5x-y^2)-\frac{\partial}{\partial y}(-4xy)\big)\,\mathrm{d}A$$

$$=\iint_A(5+4x)\,\mathrm{d}A=\int_{-1}^{1}\int_{3x^2}^{3}(5+4x)\ \mathrm{d}y\ \mathrm{d}x$$

$$=\int_{-1}^{1}\Big/_{3x^2}^{3}(5y+4xy)\ \mathrm{d}x=\int_{-1}^{1}(15+12x-15x^2-12x^3)\ \mathrm{d}x$$

$$=\Big/_{-1}^{1}(15x+6x^2-5x^3-3x^4)=20.$$

b) Greenin lauseen avulla saadaan

$$\oint_{\partial A}(xy^2-3x^2)\ \mathrm{d}x+(3xy+7y)\ \mathrm{d}y=\iint_A\big[\frac{\partial(3xy+7y)}{\partial x}-\frac{\partial(xy^2-3x^2)}{\partial y}\big]\ \mathrm{d}A$$

$$=\iint_A(3y-2xy)\,\mathrm{d}A=\int_0^2\int_{-1}^{\frac{1}{2}y}(3y-2xy)\ \mathrm{d}x\ \mathrm{d}y=\int_0^2\Big/_{-1}^{\frac{1}{2}y}(3yx-yx^2)\ \mathrm{d}y$$

$$=\int_0^2(\frac{3}{2}y^2-\frac{1}{4}y^3+3y+y)\ \mathrm{d}y=\int_0^2(\frac{3}{2}y^2-\frac{1}{4}y^3+4y)\ \mathrm{d}y$$

$$=\Big/_0^2(\frac{1}{2}y^3-\frac{1}{16}y^4+2y^2)=4-1+8=11.$$

TAI

$$\oint_{\partial A}(xy^2-3x^2)\ \mathrm{d}x+(3xy+7y)\ \mathrm{d}y=\iint_A(3y-2xy)\,\mathrm{d}A$$

$$=\int_{-1}^{0}\int_0^2(3y-2xy)\ \mathrm{d}y\ \mathrm{d}x+\int_0^1\int_{2x}^{2}(3y-2xy)\ \mathrm{d}y\ \mathrm{d}x$$

$$=\int_{-1}^{0}\Big/_0^2(\frac{3}{2}y^2-xy^2)\ \mathrm{d}x+\int_0^1\Big/_{2x}^{2}(\frac{3}{2}y^2-xy^2)\ \mathrm{d}x$$

$$=\int_{-1}^{0}(6-4x)\ \mathrm{d}x+\int_0^1(6-4x-6x^2+4x^3)\ \mathrm{d}x$$

$$=\Big/_{-1}^{0}(6x-2x^2)+\Big/_0^1(6x-2x^2-2x^3+x^4)$$

$$=6+2+6-2-2+1=11$$

c)

$$\oint_{\partial A}(-8x^2y)\ \mathrm{d}x + 8xy^2\ \mathrm{d}y = \iint_A \Big(\frac{\partial}{\partial x}(8xy^2) - \frac{\partial}{\partial y}(-8x^2y)\Big)\,\mathrm{d}A = \iint_A (8y^2+8x^2)\,\mathrm{d}A$$

$$= \int_{\frac{\pi}{2}}^{\pi}\int_0^5 8r^2\,r\ \mathrm{d}r\,\mathrm{d}\varphi = \int_{\frac{\pi}{2}}^{\pi}\int_0^5 8r^3\ \mathrm{d}r\,\mathrm{d}\varphi = \int_{\frac{\pi}{2}}^{\pi}\Big(\Big/_0^5\ 2r^4\Big)\,\mathrm{d}\varphi$$

$$= \int_{\frac{\pi}{2}}^{\pi} 1250\,\mathrm{d}\varphi = 625\pi$$

d) Greenin lauseen nojalla

$$\oint_{\partial A}\vec{F}\cdot\mathrm{d}\vec{x} = \iint_A (Q_x - P_y)\,\mathrm{d}A = \int_0^1\int_0^{4x^3}(18xy-14xy)\ \mathrm{d}y\ \mathrm{d}x$$

$$= \int_0^1 \Big/_0^{4x^3}\ 2xy^2\ \mathrm{d}x = \int_0^1 32x^7\ \mathrm{d}x = \Big/_0^1\ 4x^8 = 4.$$

Vektorikenttä ei ole konservatiivinen, sillä yllä olevan käyräintegraalin arvo suljetun käyrän yli eroaa nollasta.

14. a) Koska $f_x(x,y)=2x$ ja $f_y(x,y)=2y$, niin

$$\iint_S \sqrt{4z-11}\,\mathrm{d}S = \iint_A \sqrt{4(x^2+y^2+3)-11}\sqrt{1+(2x)^2+(2y)^2}\,\mathrm{d}A$$

$$= \int_0^1\int_0^3 (1+4x^2+4y^2)\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 \Big/_0^3 (y+4x^2y+\frac{4}{3}y^3)\ \mathrm{d}x$$

$$= \int_0^1 (39+12x^2)\ \mathrm{d}x = \Big/_0^1 (39x+4x^3) = 43.$$

b) Koska $f_x(x,y)=-2x$ ja $f_y(x,y)=-2y$, niin

$$\iint_S 6\sqrt{17-4z}\,\mathrm{d}S = \iint_A 6\sqrt{17-4(4-x^2-y^2)}\sqrt{1+(-2x)^2+(-2y)^2}\,\mathrm{d}A$$

$$= \int_0^1\int_0^x (6+24x^2+24y^2)\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 \Big/_0^x (6y+24x^2y+8y^3)\ \mathrm{d}x$$

$$= \int_0^1 (6x+32x^3)\ \mathrm{d}x = \Big/_0^1 (3x^2+8x^4) = 11.$$

15. a) Annetun avoimen funktiopinnan ulkoinen yksikkönormaalivektori

$$\vec{n}^0 = \frac{-f_x(x,y)\vec{i} - f_y(x,y)\vec{j} + \vec{k}}{\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}} = \frac{-3y\vec{i}-3x\vec{j}+\vec{k}}{\sqrt{1+9x^2+9y^2}}.$$

$$\begin{aligned}
\text{vuo} &= \iint_S \vec{F}\cdot\vec{n}^0\,\mathrm{d}S = \iint_S (2x,-6y,4z-28)\cdot\frac{(-3y,-3x,1)}{\sqrt{1+9x^2+9y^2}}\,\mathrm{d}S \\
&= \iint_S \frac{-6xy+18xy+4z-28}{\sqrt{1+9x^2+9y^2}}\,\mathrm{d}S = \iint_S \frac{12xy+4z-28}{\sqrt{1+9x^2+9y^2}}\,\mathrm{d}S \\
&= \iint_A \frac{12xy+4(7+3xy)-28}{\sqrt{1+9x^2+9y^2}}\sqrt{1+9x^2+9y^2}\,\mathrm{d}A = \iint_A 24xy\,\mathrm{d}A \\
&= \int_0^1\int_0^1 24xy\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 \Big/_0^1\ 12xy^2\ \mathrm{d}x = \int_0^1 12x\,dx = \Big/_0^1\ 6x^2 = 6.
\end{aligned}$$

b) Annetun avoimen funktiopinnan ulkoinen yksikkönormaalivektori

$$\vec{n}^0 = \frac{-f_x(x,y)\vec{i}-f_y(x,y)\vec{j}+\vec{k}}{\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}} = \frac{-3y\vec{i}-3x\vec{j}+\vec{k}}{\sqrt{1+9x^2+9y^2}}.$$

$$\begin{aligned}
\text{vuo} &= \iint_S \vec{F}\cdot\vec{n}^0\,\mathrm{d}S = \iint_S (-4xz,4yz-3y,z-1)\cdot\frac{(-3y,-3x,1)}{\sqrt{1+9x^2+9y^2}}\,\mathrm{d}S \\
&= \iint_S \frac{12xyz-12xyz+9xy+z-1}{\sqrt{1+9x^2+9y^2}}\,\mathrm{d}S = \iint_A \frac{9xy+3xy+1-1}{\sqrt{1+9x^2+9y^2}}\sqrt{1+9x^2+9y^2}\,\mathrm{d}A \\
&= \iint_A 12xy\,\mathrm{d}A = \int_0^1\int_0^{2x} 12xy\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 \Big/_0^{2x}\ 6xy^2\ \mathrm{d}x \\
&= \int_0^1 24x^3\ \mathrm{d}x = \Big/_0^1\ 6x^4 = 6
\end{aligned}$$

c) Annetun avoimen funktiopinnan ulkoinen yksikkönormaalivektori

$$\vec{n}^0 = \frac{-f_x(x,y)\vec{i}-f_y(x,y)\vec{j}+\vec{k}}{\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}} = \frac{2x\vec{i}+2y\vec{j}+\vec{k}}{\sqrt{1+4x^2+4y^2}}.$$

Lasketaan vuo:

$$\begin{aligned}
\text{vuo} &= \iint_S \vec{F}\cdot\vec{n}^0\,\mathrm{d}S = \iint_S (5y-3yz,-5x+3xz,6x-xz)\cdot\frac{(2x,2y,1)}{\sqrt{1+4x^2+4y^2}}\,\mathrm{d}S \\
&= \iint_S \frac{10xy-6xyz-10xy+6xyz+6x-xz}{\sqrt{1+4x^2+4y^2}}\,\mathrm{d}S = \iint_S \frac{6x-xz}{\sqrt{1+4x^2+4y^2}}\,\mathrm{d}S \\
&= \iint_A \frac{6x-x(6-x^2-y^2)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+4x^2+4y^2}\,\mathrm{d}A = \iint_A (x^3+xy^2)\,\mathrm{d}A \\
&= \int_0^1\int_0^{2x}(x^3+xy^2)\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 \Big/_0^{2x}\ (x^3y+\frac{1}{3}xy^3)\ \mathrm{d}x = \int_0^1 (2x^4+\frac{8}{3}x^4)\ \mathrm{d}x \\
&= \int_0^1 \frac{14}{3}x^4\ \mathrm{d}x = \Big/_0^1\ \frac{14}{15}x^5 = \frac{14}{15}.
\end{aligned}$$

d) Annetun avoimen funktiopinnan ulkoinen yksikkönormaalivektori

$$\vec{n}^0 = \frac{(-f_x(x,y), -f_y(x,y), 1)}{\sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}} = \frac{(-2y, -2x, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 5x\}$$

$$\begin{aligned}
\text{vuo} &= \iint_S \vec{F} \cdot \vec{n}^0 \, \mathrm{d}S = \iint_S (3x - 5xz, 5yz - 8y, 3z - 51) \cdot \frac{(-2y, -2x, 1)}{\sqrt{1 + 4x^2 + 4y^2}} \, \mathrm{d}S \\
&= \iint_S \frac{-6xy + 10xyz - 10xyz + 16xy + 3z - 51}{\sqrt{1 + 4x^2 + 4y^2}} \, \mathrm{d}S = \iint_S \frac{10xy + 3z - 51}{\sqrt{1 + 4x^2 + 4y^2}} \, \mathrm{d}S \\
&= \iint_A \frac{10xy + 3(2xy + 17) - 51}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2} \, \mathrm{d}A = \iint_A 16xy \, \mathrm{d}A \\
&= \int_0^1 \int_0^{5x} 16xy \ \mathrm{d}y \ \mathrm{d}x = \int_0^1 \Big/_0^{5x} 8xy^2 \ \mathrm{d}x = \int_0^1 200x^3 \ \mathrm{d}x = \Big/_0^1 50x^4 = 50
\end{aligned}$$

16. a) Koska $\vec{F}(x,y,z) = (5y + 3z^2)\vec{i} + (5x + 4z^2)\vec{j} + (6xz + 2yz)\vec{k}$, niin

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 5y + 3z^2 & 5x + 4z^2 & 6xz + 2yz \end{vmatrix} \\
&= (2z - 8z)\vec{i} - (6z - 6z)\vec{j} + (5 - 5)\vec{k} = -6z\vec{i}.
\end{aligned}$$

Stokesin lauseen mukaan

$$\begin{aligned}
&\oint_{\partial S} (5y + 3z^2) \ \mathrm{d}x + (5x + 4z^2) \ \mathrm{d}y + (6xz + 2yz) \ \mathrm{d}z = \oint_{\partial S} \vec{F} \cdot d\vec{x} \overset{\text{Stokes}}{=} \iint_S \nabla \times \vec{F} \cdot \vec{n}^0 \, \mathrm{d}S \\
&= \iint_S (-6z, 0, 0) \cdot \frac{(-y, -x, 1)}{\sqrt{1 + x^2 + y^2}} \, \mathrm{d}S = \iint_S \frac{6yz}{\sqrt{1 + x^2 + y^2}} \, \mathrm{d}S \\
&= \iint_A \frac{6y(xy)}{\sqrt{1 + x^2 + y^2}} \sqrt{1 + x^2 + y^2} \, \mathrm{d}A \\
&= \int_1^2 \int_1^2 6xy^2 \ \mathrm{d}y \ \mathrm{d}x = \int_1^2 \Big/_1^2 2xy^3 \ \mathrm{d}x = \int_1^2 2x(8 - 1) \ \mathrm{d}x \\
&= \int_1^2 14x \ \mathrm{d}x = \Big/_1^2 7x^2 = 21.
\end{aligned}$$

b) Koska $\vec{F}(x,y,z) = (6y^2 + 7xyz)\vec{i} + 12xy\vec{j} + 7xz\vec{k}$, niin

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 6y^2 + 7xyz & 12xy & 7xz \end{vmatrix} \\
&= (0 - 0)\vec{i} - (7z - 7xy)\vec{j} + (12y - 12y - 7xz)\vec{k} = (7xy - 7z)\vec{j} - 7xz\vec{k}.
\end{aligned}$$

Stokesin lauseen mukaan

$$\oint_{\partial S}(6y^2+7xyz)\ \mathrm{d}x+12xy\ \mathrm{d}y+7xz\ \mathrm{d}z=\oint_{\partial S}\vec{F}\cdot\mathrm{d}\vec{x}\overset{\text{Stokes}}{=}\iint_S\nabla\times\vec{F}\cdot\vec{n}^0\,\mathrm{d}S$$
$$=\iint_S(0,7xy-7z,-7xz)\cdot\frac{(-2y,-2x,1)}{\sqrt{1+4x^2+4y^2}}\,\mathrm{d}S$$
$$=\iint_S\frac{7x(-2xy+2z-z)}{\sqrt{1+4x^2+4y^2}}\,\mathrm{d}S$$
$$=\iint_A\frac{7x(-2xy+(15+2xy))}{\sqrt{1+4x^2+4y^2}}\sqrt{1+4x^2+4y^2}\,\mathrm{d}A$$
$$=\int_0^1\int_{-5x}^{2\sqrt{x}}105x\ \mathrm{d}y\ \mathrm{d}x=\int_0^1\Big/_{-5x}^{2\sqrt{x}}105xy\ \mathrm{d}x$$
$$=\int_0^1 105x(2x^{\frac{1}{2}}+5x)\ \mathrm{d}x=\int_0^1 105(2x^{\frac{3}{2}}+5x^2)\ \mathrm{d}x$$
$$=\Big/_0^1 105(\frac{4}{5}x^{\frac{5}{2}}+\frac{5}{3}x^3)$$
$$=84+175=259.$$

17. a) Lasketaan vuo:

$$\iint_S\vec{F}\cdot\vec{n}^0\,\mathrm{d}S=\iiint_V\nabla\cdot\vec{F}\,\mathrm{d}V$$
$$=\iiint_V(\partial_x,\partial_y,\partial_z)\cdot(7xz-3x+4,3y-7yz-2,8xyz^2)\,\mathrm{d}V$$
$$=\iiint_V(7z-3+3-7z+16xyz)\,\mathrm{d}V=\iiint_V 16xyz\,\mathrm{d}V=\int_0^3\int_0^x\int_{-1}^2 16xyz\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x$$
$$=\int_0^3\int_0^x\Big/_{-1}^2 8xyz^2\ \mathrm{d}y\ \mathrm{d}x=\int_0^3\int_0^x 8xy(4-1)\ \mathrm{d}y\ \mathrm{d}x=\int_0^3\int_0^x 24xy\ \mathrm{d}y\ \mathrm{d}x$$
$$=\int_0^3\Big/_0^x 12xy^2\ \mathrm{d}x=\int_0^3 12x^3\ \mathrm{d}x=\Big/_0^3 3x^4=243.$$

b)

$$\iint_S\vec{F}\cdot\vec{n}^0\,\mathrm{d}S=\iiint_V\nabla\cdot\vec{F}\,\mathrm{d}V$$
$$=\iiint_V(\partial_x,\partial_y,\partial_z)\cdot(-4x+y,5x+4y,4z^3-2xy)\,\mathrm{d}V$$
$$=\iiint_V(-4+4+12z^2)\,\mathrm{d}V=\iiint_V 12z^2\,\mathrm{d}V$$

Siirrytään sylinterikoordinaatistoon:

$$\iiint_V 12z^2 \,\mathrm{d}V = \iiint_{V'} 12z^2\, r \;\mathrm{d}z \;\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 12z^2\, r \;\mathrm{d}z \;\mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^2 \Big/_0^{4-r^2} 4rz^3 \;\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^2 4r[(4-r^2)^3 - 0] \;\mathrm{d}r\,\mathrm{d}\varphi$$

$$= -2\int_0^{2\pi} \Big/_0^2 \frac{(4-r^2)^4}{4} \,\mathrm{d}\varphi = -2\int_0^{2\pi} (-64)\,\mathrm{d}\varphi = 256\pi.$$

c)

$$\text{vuo} = \iint_S \vec{F}\cdot\vec{n}^0 \,\mathrm{d}S = \iiint_V \nabla\cdot\vec{F}\,\mathrm{d}V$$

$$= \iiint_V (\partial_x, \partial_y, \partial_z)\cdot(6x - yz, 3xz - 6y, z^4 - 5xy)\,\mathrm{d}V$$

$$= \iiint_V (6 - 6 + 4z^3)\,\mathrm{d}V = \iiint_V 4z^3\,\mathrm{d}V$$

Siirrytään sylinterikoordinaatistoon, jossa

$$V' = \{(r,\varphi,z)\in\mathbb{R}^3 : 0\le r\le 2, 0\le\varphi\le 2\pi, r^2-4\le z\le 0\}.$$

Siis

$$\iiint_V 4z^3\,\mathrm{d}V = \iiint_{V'} 4z^3\, r \;\mathrm{d}z \;\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^2\int_{r^2-4}^{0} 4z^3\, r \;\mathrm{d}z \;\mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^2 \Big/_{r^2-4}^{0} rz^4 \;\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^2 [-r(r^2-4)^4] \;\mathrm{d}r\,\mathrm{d}\varphi$$

$$= \int_0^{2\pi} \Big/_0^2 [-\frac{(r^2-4)^5}{10}]\,\mathrm{d}\varphi = \int_0^{2\pi} (-\frac{4^5}{10})\,\mathrm{d}\varphi = -\frac{1024\pi}{5}.$$