031075P MATEMATIIKAN PERUSKURSSI II

Ratkaisut 1. välikokeeseen 23.1.2018

1. a) Osoita sarja $\sum_{k=1}^{\infty} \dfrac{4k}{4k^3+5}$ suppenevaksi vertailuperiaatteen avulla.

b) Tutki sarjan

$$\sum_{k=1}^{\infty} \left(\frac{1}{4k-1} - \frac{1}{4k+3} \right)$$

suppenemista osasummien jonon $(S_n)_{n=1}^{\infty}$, missä

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{4k-1} - \frac{1}{4k+3} \right), n = 1, 2, 3, \ldots ,$$

avulla.

c) Laske potenssisarjojen avulla raja-arvo

$$\lim_{x \to 0} \frac{x \,\overline{\text{arc}}\tan(x^3) - \sinh(x^4)}{\cosh(x^5) - 1}.$$

1. a) Vertailuperiaate:

$$0 < \frac{4k}{4k^3+5} < \frac{4k}{4k^3} = \frac{1}{k^2}$$

kaikilla $k = 1,\ 2,\ 3,\ \ldots$. Vertailusarja $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ suppenee ($p = 2 > 1$), joten tutkittava sarja suppenee vertailuperiaatteen nojalla.

b) Muodostetaan osasummien jono

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{1}{4k-1} - \frac{1}{4k+3} \right) \\ &= \left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{11} \right) + \cdots + \left(\frac{1}{4n-5} - \frac{1}{4n-1} \right) + \left(\frac{1}{4n-1} - \frac{1}{4n+3} \right) \\ &= \frac{1}{3} - \frac{1}{4n+3},\ n = 1, 2, \ldots , \end{aligned}$$

josta saadaan

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{1}{3} - \frac{1}{4n+3} \right) = \frac{1}{3}.$$

Sarja siis suppenee.

c) Tunnetaan potenssisarjat

$$\overline{\text{arc}}\tan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,\ |x| < 1,$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots,\ x \in \mathbb{R},$$
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots,\ x \in \mathbb{R},$$

joten saadaan

$$\lim_{x\to 0} \frac{x\,\overline{\text{arc}}\tan(x^3) - \sinh(x^4)}{\cosh(x^5) - 1}$$
$$= \lim_{x\to 0} \frac{x(x^3 - \frac{(x^3)^3}{3} + \frac{(x^3)^5}{5} - \cdots) - (x^4 + \frac{(x^4)^3}{3!} + \frac{(x^4)^5}{5!} + \cdots)}{1 + \frac{(x^5)^2}{2!} + \frac{(x^5)^4}{4!} + \frac{(x^5)^6}{6!} + \cdots - 1}$$
$$= \lim_{x\to 0} \frac{x(x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \cdots) - (x^4 + \frac{x^{12}}{3!} + \frac{x^{20}}{5!} + \cdots)}{\frac{x^{10}}{2!} + \frac{x^{20}}{4!} + \frac{x^{30}}{6!} + \cdots}$$
$$= \lim_{x\to 0} \frac{x^4 - \frac{x^{10}}{3} + \frac{x^{16}}{5} - \cdots - x^4 - \frac{x^{12}}{3!} - \frac{x^{20}}{5!} - \cdots}{\frac{x^{10}}{2!} + \frac{x^{20}}{4!} + \frac{x^{30}}{6!} + \cdots}$$
$$= \lim_{x\to 0} \frac{(-\frac{x^{10}}{3} + \frac{x^{16}}{5} - \cdots - \frac{x^{12}}{3!} - \frac{x^{20}}{5!} - \cdots) : x^{10}}{(\frac{x^{10}}{2!} + \frac{x^{20}}{4!} + \frac{x^{30}}{6!} + \cdots) : x^{10}}$$
$$= \lim_{x\to 0} \frac{-\frac{1}{3} + \frac{x^6}{5} - \cdots - \frac{x^2}{3!} - \frac{x^{10}}{5!} - \cdots}{\frac{1}{2!} + \frac{x^{10}}{4!} + \frac{x^{20}}{6!} + \cdots} = \frac{-\frac{1}{3}}{\frac{1}{2!}} = -\frac{2}{3}.$$