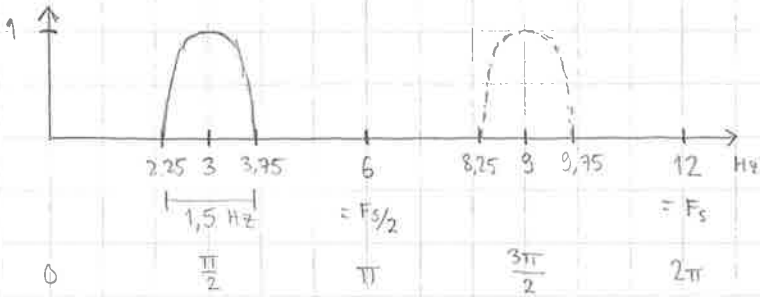
DIGITAL FILTERS – WEEK EXAM 5

PROBLEM 1



LET'S SELECT $3\text{ Hz} <=> \frac{\pi}{2}$

$\rightarrow F_s = 2\pi = 4 \cdot \frac{\pi}{2} = 4 \cdot 3\text{Hz} = 12\text{ Hz}$

SAMPLING INTERVAL 0,75 Hz

$\rightarrow N = \frac{12\text{ Hz}}{0{,}75\text{ Hz}} = 16$

$3\text{ Hz} <=> \frac{\pi}{3}$; $F_s = 18\text{ Hz}$; $N = 24$

$3\text{ Hz} <=> \frac{\pi}{4}$; $F_s = 24\text{ Hz}$; $N = 32$

ETC.

NON-ZERO SAMPLES:

$H\left(\frac{3}{0{,}75}\right) = H(4) = e^{-j\frac{\pi}{4}}$

$H\left(\frac{9}{0{,}75}\right) = H(12) = e^{j\frac{\pi}{4}}$

COMPLEX CONJUGATES → REAL VALUED COEFFICIENTS

$H(k) = 0$ IF $k \neq 4$ OR $k \neq 12$

$$H(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{j2\pi k/N} z^{-1}} = \frac{1-z^{-16}}{16} \sum_{k=0}^{15} \frac{H(k)}{1-e^{j\pi k/8} z^{-1}}$$

$$= \frac{1-z^{-16}}{16} \left(\frac{H(4)}{1-e^{j\pi 4/8} z^{-1}} + \frac{H(12)}{1-e^{j\pi 12/8} z^{-1}} \right) \quad \left| \begin{array}{l} H(4) \text{ AND } H(12) \\ \text{ARE NON-ZERO} \end{array} \right.$$

$$= \frac{1-z^{-16}}{16} \left(\frac{e^{-j\frac{\pi}{4}}}{1-e^{j\frac{\pi}{2}} z^{-1}} + \frac{e^{j\frac{\pi}{4}}}{1-e^{j\frac{3\pi}{2}} z^{-1}} \right) \quad \left| \begin{array}{l} e^{j\frac{\pi}{2}} = j \\ e^{j\frac{3\pi}{2}} = -j \end{array} \right.$$

$$= \frac{1-z^{-16}}{16} \left(\frac{e^{-j\frac{\pi}{4}}\left(1-e^{j\frac{3\pi}{2}} z^{-1}\right) + e^{j\frac{\pi}{4}}\left(1-e^{j\frac{\pi}{2}} z^{-1}\right)}{\left(1-jz^{-1}\right)\left(1+jz^{-1}\right)} \right)$$

$$= \frac{1-z^{-16}}{16} \left(\frac{e^{-j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}} \cdot e^{j\frac{6\pi}{4}} z^{-1} + e^{j\frac{\pi}{4}} - e^{j\frac{\pi}{4}} e^{j\frac{2\pi}{4}} z^{-1}}{1+z^{-2}} \right)$$

$$e^{-j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}} e^{j\frac{6\pi}{4}} z^{-1} + e^{j\frac{\pi}{4}} - e^{j\frac{\pi}{4}} e^{j\frac{2\pi}{4}} z^{-1} = e^{-j\frac{\pi}{4}} + e^{j\frac{\pi}{4}} - e^{j\frac{5\pi}{4}} z^{-1} - e^{j\frac{3\pi}{4}} z^{-1}$$

$$= \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) z^{-1} - \left(-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) z^{-1}$$

$$= 2 \cdot \frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2} z^{-1} = \sqrt{2}\left(1+z^{-1}\right)$$

$$= \frac{1-z^{-16}}{16} \cdot \frac{\sqrt{2}\left(1+z^{-1}\right)}{1+z^{-2}} = \frac{\sqrt{2}}{16} \cdot \frac{\left(1-z^{-16}\right)\left(1+z^{-1}\right)}{1+z^{-2}}$$

$$= \frac{\sqrt{2}}{16} \cdot \frac{1+z^{-1}-z^{-16}-z^{-17}}{1+z^{-2}}$$

PROBLEM 2

LET'S DECIDE $3\,Hz <=> \frac{\pi}{4}$ → $6\,Hz <=> \frac{\pi}{2}$

→ $F_s = 2\pi = 8 \cdot \frac{\pi}{4} = 8 \cdot 3\,Hz = 24\,Hz$; SAMPLING IMTERVAL $0{,}75\,Hz$

→ $N = \frac{24\,Hz}{0{,}75\,Hz} = 32$

$3\,Hz <=> \frac{\pi}{3}$; $6\,Hz <=> \frac{2\pi}{3}$; $F_s = 2\pi = 6 \cdot \frac{\pi}{3} = 6 \cdot 3\,Hz = 18\,Hz$; $N = \frac{18}{0{,}75} = 24$

NON-ZERO SAMPLES: $H(4)$, $H(8)$, $H(16)$, $H(20)$ ETC ...

NON-ZERO SAMPLES:

$H\left(\frac{3}{0{,}75}\right) = H(4)$; $H\left(\frac{6}{0{,}75}\right) = H(8)$; $H\left(\frac{18}{0{,}75}\right) = H(24)$; $H\left(\frac{21}{0{,}75}\right) = H(28)$

LINEAR PHASE AND ANTISYMMETRY (ASSUME): $\left[\frac{\pi}{4}, \frac{\pi}{2}, 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}, 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}\right]$

$\phi(\omega) = \frac{\pi}{2} - \frac{N-1}{2}\omega = \frac{\pi}{2} - \frac{31}{2}\omega$

$\phi\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{31}{2} \cdot \frac{\pi}{4} = -\frac{27}{8}\pi$ $\qquad H(4) = e^{-j\frac{27}{8}\pi} = -\frac{\sqrt{2-\sqrt{2}}}{2} + j\frac{\sqrt{\sqrt{2}+2}}{2} \approx -0{,}383 + j0{,}924$

$\phi\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{31}{2} \cdot \frac{\pi}{2} = -\frac{29}{4}\pi$ $\qquad H(8) = e^{-j\frac{29}{4}\pi} = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$ [COMPLEX CONJUGATES]

$\phi\left(\frac{3\pi}{2}\right) = \frac{\pi}{2} - \frac{31}{2} \cdot \frac{3\pi}{2} = -\frac{91}{4}\pi$ $\qquad H(24) = e^{-j\frac{91}{4}\pi} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$

$\phi\left(\frac{7\pi}{4}\right) = \frac{\pi}{2} - \frac{31}{2} \cdot \frac{7\pi}{4} = -\frac{213}{8}\pi$ $\qquad H(27) = e^{-j\frac{213}{8}\pi} = -\frac{\sqrt{2-\sqrt{2}}}{2} - j\frac{\sqrt{\sqrt{2}+2}}{2} \approx -0{,}383 - j0{,}924$

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{j2\pi nk/N} = \frac{1}{32}\sum_{k=0}^{31} H(k) e^{jnk/16}$$

$$h(0) = \frac{1}{32}\sum_{k=0}^{31} H(k) \underbrace{e^{j\pi \cdot 0 \cdot k/16}}_{=1} = \frac{1}{32}\left(e^{-j\frac{27}{8}\pi} + e^{-j\frac{29}{4}\pi} + e^{-j\frac{91}{4}\pi} + e^{-j\frac{213}{8}\pi}\right)$$

$$= \frac{1}{32}\left(2 \cdot \left(-\frac{\sqrt{2-\sqrt{2}}}{2}\right) + 2 \cdot \left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{1}{32}\left(-\sqrt{2-\sqrt{2}} - \sqrt{2}\right)$$

$$= -0{,}068112$$

IMAGINARY PARTS CANCEL OUT

$$h(1) = \frac{1}{32}\sum_{k=0}^{31} H(k) e^{j\pi k/16} = \frac{1}{32}\left(e^{-j\frac{27}{8}\pi} \cdot e^{j\frac{4\pi}{16}} + e^{-j\frac{29}{4}\pi} \cdot e^{j\frac{8\pi}{16}} + e^{-j\frac{91}{4}\pi} \cdot e^{j\frac{24\pi}{16}} + e^{-j\frac{213}{8}\pi} \cdot e^{j\frac{28\pi}{16}}\right)$$

$e^{-j\frac{27}{8}\pi} \cdot e^{j\frac{4}{16}\pi} = e^{-j\frac{50}{16}\pi} = -\frac{\sqrt{\sqrt{2}+2}}{2} + j\frac{\sqrt{2-\sqrt{2}}}{2}$

$e^{-j\frac{29}{4}\pi} \cdot e^{j\frac{8}{16}\pi} = e^{-j\frac{108}{16}\pi} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$

$e^{-j\frac{91}{4}\pi} \cdot e^{j\frac{24}{16}\pi} = e^{-j\frac{340}{16}\pi} = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$

$e^{-j\frac{213}{8}\pi} \cdot e^{j\frac{28}{16}\pi} = e^{-j\frac{398}{16}\pi} = -\frac{\sqrt{\sqrt{2}+2}}{2} - j\frac{\sqrt{2-\sqrt{2}}}{2}$

$$= \frac{1}{32}\left(2 \cdot \left(-\frac{\sqrt{\sqrt{2}+2}}{2}\right) + 2 \cdot \left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{1}{32}\left(-\sqrt{\sqrt{2}+2} - \sqrt{2}\right)$$

$$= -0{,}101937$$