DIGITAL FILTERS - WEEK EXAM 4

PROBLEM 1

LINEAR PHASE FILTER → POSITIVE OR NEGATIVE SYMMETRY

$H_1(z) = 1 - 5z^{-1} + 20z^{-2} + 20z^{-3} + 5z^{-4} - z^{-5}$ NO SYMMETRY

$h_1(n) = \{1 \ -5 \ 20 \ \vdots \ 20 \ 5 \ -1\}$ NOT LINEAR PHASE

$H_2(z) = 1 - 3z^{-1} + 18z^{-2} + 18z^{-3} - 3z^{-4} + z^{-5}$ POSITIVE SYMMETRY

$h_2(n) = \{1 \ -3 \ 18 \ \vdots \ 18 \ -3 \ 1\}$ EVEN N → TYPE 2

$H_3(z) = 1 - 5z^{-1} + 20z^{-2} + z^{-3} - 20z^{-4} + 5z^{-5} - z^{-6}$ NO SYMMETRY

$h_3(n) = \{1 \ -5 \ 20 \ 1 \ -20 \ 5 \ -1\}$ NOT LINEAR PHASE

$H_4(z) = 1 - 5z^{-1} + 20z^{-2} - 20z^{-4} + 5z^{-5} - z^{-6}$ NEGATIVE SYMMETRY

$h_4(n) = \{1 \ -5 \ 20 \ 0 \ -20 \ 5 \ -1\}$ ODD N → TYPE 3

⇒ SUITABLE FILTERS: $H_2(z)$ AND $H_4(z)$

LET'S CHOOSE $H_2(z)$: (TYPE 2, $N=6$)

TYPE 2 FILTER: $\phi(\omega) = -\frac{N-1}{2}\omega = -\frac{6-1}{2}\omega = -\frac{5}{2}\omega$

MAGNITUDE RESPONSE: ($\frac{\pi}{2}$ INTERVALS)

Euler's formula
$e^{j\omega} = \cos\omega + j \cdot \sin\omega$
e.g.
$e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\cdot\sin\frac{\pi}{2}$
$= 0 - j \cdot 1 = -j$

$H_2(e^{j\omega}) = 1 - 3e^{-j\omega} + 18e^{-j2\omega} + 18e^{-j3\omega} - 3e^{-j4\omega} + e^{-j5\omega}$

$H_2(e^{j0}) = 1 - 3 + 18 + 18 - 3 + 1 = 32$; $20\log 32 = 30{,}10$ dB

$$H_2(e^{j\frac{\pi}{2}}) = 1 - 3e^{-j\frac{\pi}{2}} + 18e^{-j\pi} + 18e^{-j\frac{3\pi}{2}} - 3e^{-j2\pi} + e^{-j\frac{5\pi}{2}}$$
$$= 1 - 3\cdot(-j) + 18\cdot(-1) + 18\cdot j - 3\cdot 1 + (-j)$$
$$= 1 + j3 - 18 + j18 - 3 - j = -20 + j20$$

$20\log_{10}\sqrt{(-20)^2 + 20^2} = 29{,}03$ dB

$$H_2(e^{j\pi}) = 1 - 3e^{-j\pi} + 18e^{-j2\pi} + 18e^{-j3\pi} - 3e^{-j4\pi} + e^{-j5\pi}$$
$$= 1 - 3\cdot(-1) + 18\cdot 1 + 18\cdot(-1) - 3\cdot 1 + 1\cdot(-1) = 0 \Rightarrow -\infty \text{ dB}$$

SYMMETRY: $H_2(e^{j\frac{3\pi}{2}}) = -20 - j20$ (29,03 dB)

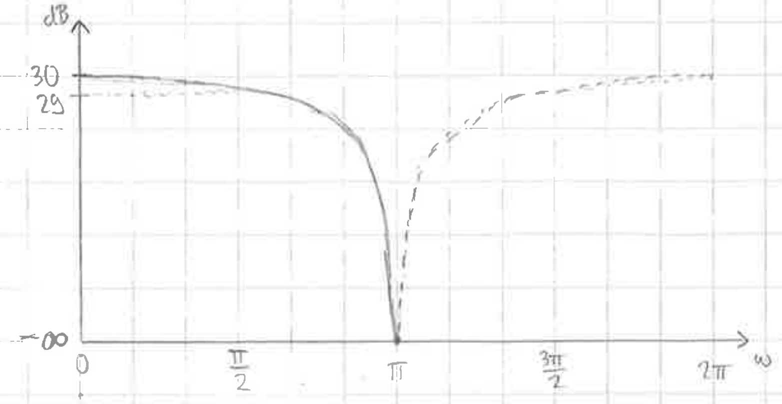
$H_2(e^{j2\pi}) = 32$ (30,10 dB)

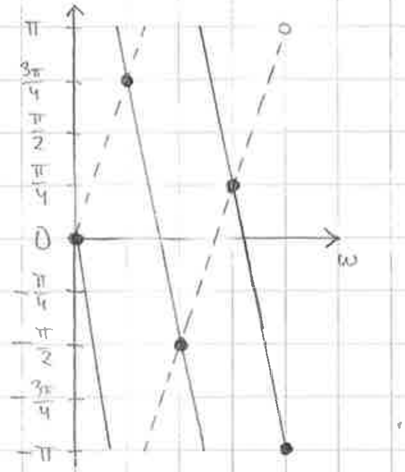
PHASE RESPONSE: $\phi(0) = 0$; $\phi(\frac{\pi}{2}) = -\frac{5\pi}{4}$ $(= \frac{3\pi}{4})$; $\phi(\pi) = -\frac{5\pi}{2}$ $(= -\frac{\pi}{2})$

$\phi(\frac{3\pi}{2}) = -\frac{15\pi}{4}$ $(= \frac{\pi}{4})$; $\phi(2\pi) = -5\pi$ $(= \pi)$

WRAPPED PHASE
$\phi(\omega) \pm k\cdot 2\pi$

$H_2(z)$: $\|H(\omega)\| = \{\, 30{,}10 \quad 29{,}03 \quad -\infty \quad 29{,}03 \quad 30{,}10 \,\}$ dB
$\phi(\omega) = \{\, 0 \quad \frac{3\pi}{4} \quad -\frac{\pi}{2} \quad \frac{\pi}{4} \quad \pi \,\}$





$H_4(z)$: (TYPE 3, $N = 7$) $\quad \phi(\omega) = \frac{\pi}{2} - \frac{N-1}{2}\omega = \frac{\pi}{2} - 3\omega$

MAGNITUDE RESPONSE ($\frac{\pi}{2}$ INTERVALS)

$$H_4(e^{j\omega}) = 1 - 5e^{-j\omega} + 20e^{-j2\omega} - 20e^{-j4\omega} + 5e^{-j5\omega} - e^{-j6\omega}$$

$$H_4(e^{j0}) = 1 - 5 + 20 - 20 + 5 - 1 = 0 \qquad (-\infty \text{ dB})$$

$$H_4(e^{j\frac{\pi}{2}}) = 1 - 5e^{-j\frac{\pi}{2}} + 20e^{-j\pi} - 20e^{-j2\pi} + 5e^{-j\frac{5\pi}{2}} - e^{-j3\pi}$$
$$= 1 - 5\cdot(-j) + 20\cdot(-1) - 20\cdot 1 + 5\cdot(-j) - 1\cdot(-1)$$
$$= 1 + j5 - 20 - 20 - j5 + 1 = -38 \qquad (20\log(|-38|) = 31{,}60 \text{ dB})$$

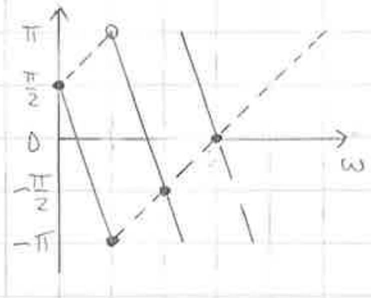
$$H_4(e^{j\pi}) = 1 - 5e^{-j\pi} + 20e^{-j2\pi} - 20e^{-j4\pi} + 5e^{-j5\pi} - e^{-j6\pi}$$
$$= 1 - 5\cdot(-1) + 20\cdot 1 - 20\cdot 1 + 5\cdot(-1) - 1\cdot 1 = 0 \qquad (-\infty \text{ dB})$$

SYMMETRY: $H_4(e^{j\frac{3\pi}{2}}) = -38$ (31,60 dB) ; $H_4(e^{j2\pi}) = 0$ ($-\infty$ dB)

PHASE RESPONSE: $\phi(0) = \frac{\pi}{2}$; $\phi(\frac{\pi}{2}) = -\pi \; (=\pi)$; $\phi(\pi) = -\frac{5\pi}{2} \; (= -\frac{\pi}{2})$
$\phi(\frac{3\pi}{2}) = -4\pi \; (=0)$; $\phi(2\pi) = -\frac{11\pi}{2} \; (= \frac{\pi}{2})$

WRAPPED PHASE
$\phi(\omega) \pm k\cdot 2\pi$
$k = 0, 1, 2, \ldots$

$H_4(z)$: $\|H(\omega)\| = \{\, -\infty \quad 31{,}6 \quad -\infty \quad 31{,}6 \quad -\infty \,\}$; $\phi(\omega) = \{\, \frac{\pi}{2} \quad \pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \,\}$



PROBLEM 2

$F_s = 40$ kHz

$f_p = 6$ kHz $\quad (= f_c)$

$f_s = 15$ kHz

$A_s > 43$ dB

$\delta_p < 0{,}0023 \quad \Rightarrow \quad A_p < 20 \log(1 + 0{,}0023) = 0{,}019955$ dB

WHICH WINDOW FUNCTIONS MEET THESE SPECIFICATIONS? (A_p AND A_s)

HANNING: $A_s = 44$ dB $\quad (> 43$ dB$) \quad$ OK!
$A_p = 0{,}0546$ dB $\quad$ NOT SMALL ENOUGH!

HAMMING: $A_s = 53$ dB $\quad (> 43$ dB$) \quad$ OK!
$A_p = 0{,}0194$ dB $\quad (< 0{,}019955$ dB$) \quad$ OK!

BLACKMAN AND KAISER ($\beta = 6{,}76$ and $\beta = 8{,}96$) ARE OK AS WELL!

HAMMING WINDOW IS NEAREST TO MEET THE SPECS

LOW-PASS FILTER USING HAMMING WINDOW

IDEAL IMPULSE RESPONSE: $\begin{cases} h_D(0) = 2f_c \\ h_D(n) = 2f_c \dfrac{\sin(n\omega_c)}{n\omega_c} \end{cases}$ $\quad$ (THESE USE NORMALIZED VALUES (f_c and ω_c))

$N = \dfrac{3{,}3}{\Delta f}$; $\quad \Delta f = \dfrac{f_s - f_p}{F_s} = \dfrac{15-6}{40} = 0{,}225 \quad \Rightarrow N = \dfrac{3{,}3}{\Delta f} = \dfrac{3{,}3}{0{,}225} = 14{,}67 \approx \underline{\underline{15}}$

NORMALIZED f_c: $\quad f_c' = \dfrac{f_c}{F_s} = \dfrac{6}{40} = 0{,}15$

$\omega_c' = 2\pi \cdot 0{,}15 = 0{,}3\pi \quad (= 0{,}94248)$

$h_D(0) = 2 \cdot 0{,}15 = 0{,}3$

$h_D(1) = 2 \cdot 0{,}15 \cdot \dfrac{\sin 0{,}3\pi}{0{,}3\pi} = 0{,}257518 \approx 0{,}258 \quad (= h_D(-1))$

WINDOW VALUES:

$w(0) = 0{,}54 + 0{,}46 \cos\Big(\overbrace{\tfrac{2\pi \cdot 0}{15}}^{=0}\Big) = 1$

$w(1) = 0{,}54 + 0{,}46 \cos\left(\dfrac{2\pi \cdot 1}{15}\right) = 0{,}960231 \approx 0{,}960 \quad (= w(-1))$

IMPULSE RESPONSE:

$h(0) = h_D(0) \cdot w(0) = 0{,}3 \cdot 1 = \underline{\underline{0{,}3}}$

$h(1) = h(-1) = h_D(1) \cdot w(1) = 0{,}258 \cdot 0{,}960 = 0{,}247277 \approx \underline{\underline{0{,}247}}$