DIGITAL FILTERS – WEEK EXAM 3

PROBLEM 1

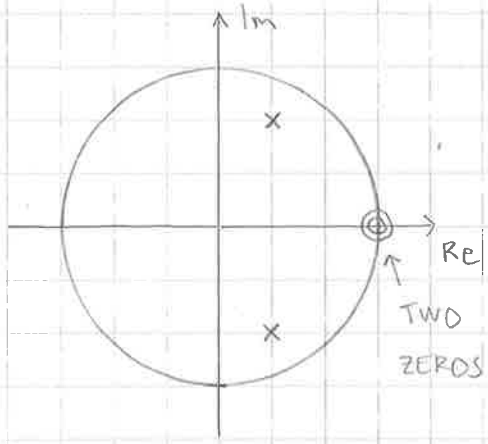
$$H(z) = \frac{z^2 - 2z + 1}{z^2 - \frac{2}{3}z + \frac{5}{9}}$$

ZEROS: $z^2 - 2z + 1 = 0$

POLES: $z^2 - \frac{2}{3}z + \frac{5}{9} = 0$

ZEROS:
$$z^2 - 2z + 1 = 0$$
$$z^2 - 2\cdot 1\cdot z + 1^2 = 0$$
$$(z-1)^2 = 0$$
$$(z_1 - 1)(z_2 - 1) = 0 \Rightarrow z_{1,2} = 1$$

ZERO-POLE DIAGRAM



POLES:
$$z^2 - \frac{2}{3}z + \frac{5}{9} = 0$$
$$z = \frac{-\left(-\frac{2}{3}\right) \pm \sqrt{\left(-\frac{2}{3}\right)^2 - 4\cdot 1\cdot \frac{5}{9}}}{2\cdot 1}$$
$$= \frac{\frac{2}{3} \pm \sqrt{\frac{4}{9} - \frac{20}{9}}}{2} = \frac{\frac{2}{3} \pm \sqrt{-\frac{16}{9}}}{2}$$
$$= \frac{\frac{2}{3} \pm \sqrt{-1\cdot\frac{16}{9}}}{2} = \frac{\frac{2}{3} \pm j\frac{4}{3}}{2}$$
$$\Rightarrow z = \frac{1}{3} \pm j\frac{2}{3}$$

$$H(e^{j\omega}) = \frac{e^{j2\omega} - 2e^{j\omega} + 1}{e^{j2\omega} - \frac{2}{3}e^{j\omega} + \frac{5}{9}}$$

FREQUENCY RESPONSE $[0, 2\pi)$

$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

$\omega = 0$:
$$H(e^{j0}) = \frac{e^{j2\cdot 0} - 2e^{j0} + 1}{e^{j2\cdot 0} - \frac{2}{3}e^{j0} + \frac{5}{9}} = \frac{1 - 2 + 1}{1 - \frac{2}{3} + \frac{5}{9}} = 0$$

CAN ALSO BE SEEN FROM ZERO-POLE DIAGRAM

$\omega = \frac{\pi}{4}$:
$$H(e^{j\frac{\pi}{4}}) = \frac{e^{j\frac{\pi}{2}} - 2e^{j\frac{\pi}{4}} + 1}{e^{j\frac{\pi}{2}} - \frac{2}{3}e^{j\frac{\pi}{4}} + \frac{5}{9}} = \frac{\cos\frac{\pi}{2} + j\sin\frac{\pi}{2} - 2\cos\frac{\pi}{4} - 2j\sin\frac{\pi}{4} + 1}{\cos\frac{\pi}{2} + j\sin\frac{\pi}{2} - \frac{2}{3}\cos\frac{\pi}{4} - \frac{2}{3}j\sin\frac{\pi}{4} + \frac{5}{9}}$$
$$= \frac{0 + j - 2\cdot\frac{\sqrt{2}}{2} - 2j\frac{\sqrt{2}}{2} + 1}{0 + j - \frac{2}{3}\cdot\frac{\sqrt{2}}{2} - \frac{2}{3}j\frac{\sqrt{2}}{2} + \frac{5}{9}} = \frac{j - \sqrt{2} - j\sqrt{2} + 1}{j - \frac{\sqrt{2}}{3} - j\frac{\sqrt{2}}{3} + \frac{5}{9}}$$
$$= \frac{1 - \sqrt{2} + j(1 - \sqrt{2})}{-\frac{\sqrt{2}}{3} + \frac{5}{9} + j\left(1 - \frac{\sqrt{2}}{3}\right)} = -0{,}8859 + j0{,}6426$$

$\omega = \frac{\pi}{2}$: $$H(e^{j\frac{\pi}{2}}) = \frac{e^{j\pi} - 2e^{j\frac{\pi}{2}} + 1}{e^{j\pi} - \frac{2}{3}e^{j\frac{\pi}{2}} + \frac{5}{9}} = \frac{\cos\pi + j\sin\pi - 2\cos\frac{\pi}{2} - 2j\sin\frac{\pi}{2} + 1}{\cos\pi + j\sin\pi - \frac{2}{3}\cos\frac{\pi}{2} - \frac{2}{3}j\sin\frac{\pi}{2} + \frac{5}{9}}$$

$$= \frac{-1 + j0 - 2\cdot 0 - 2j + 1}{-1 + j0 - \frac{2}{3}\cdot 0 - \frac{2}{3}j + \frac{5}{9}} = \frac{-2j}{-\frac{4}{9} - \frac{2}{3}j} = 2{,}0769 + j\,1{,}3846$$

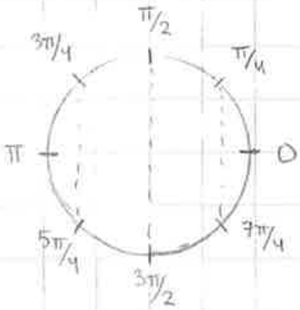
$\omega = \frac{3\pi}{4}$: $$H(e^{j\frac{3\pi}{4}}) = \frac{e^{j\frac{3\pi}{2}} - 2e^{j\frac{3\pi}{4}} + 1}{e^{j\frac{3\pi}{2}} - \frac{2}{3}e^{j\frac{3\pi}{4}} + \frac{5}{9}} = \frac{\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} - 2\cos\frac{3\pi}{4} - 2j\sin\frac{3\pi}{4} + 1}{\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} - \frac{2}{3}\cos\frac{3\pi}{4} - \frac{2}{3}j\sin\frac{3\pi}{4} + \frac{5}{9}}$$

$$= \frac{0 - j - 2\cdot\left(-\frac{\sqrt{2}}{2}\right) - 2j\cdot\frac{\sqrt{2}}{2} + 1}{0 - j - \frac{2}{3}\cdot\left(-\frac{\sqrt{2}}{2}\right) - \frac{2}{3}j\frac{\sqrt{2}}{2} + \frac{5}{9}} = \ldots = 1{,}8734 + j0{,}3333$$

$\omega = \pi$: $$H(e^{j\pi}) = \frac{e^{j2\pi} - 2e^{j\pi} + 1}{e^{j2\pi} - \frac{2}{3}e^{j\pi} + \frac{5}{9}} = \frac{\cos 2\pi + j\sin 2\pi - 2\cos\pi - 2j\sin\pi + 1}{\cos 2\pi + j\sin 2\pi - \frac{2}{3}\cos\pi - \frac{2}{3}j\sin\pi + \frac{5}{9}}$$

$$= \frac{1 + j0 - 2\cdot(-1) - 2j\cdot 0 + 1}{1 + j0 - \frac{2}{3}\cdot(-1) - \frac{2}{3}j\cdot 0 + \frac{5}{9}} = \frac{1+2+1}{1+\frac{2}{3}+\frac{5}{9}} = \frac{4}{\frac{20}{9}} = \underline{\underline{1{,}8}}$$

SO FAR WE HAVE CALCULATED THE FREQUENCY RESPONSE AT $[0, \pi]$.
THE REST CAN BE DEDUCED BASED ON SYMMETRY



COMPLEX CONJUGATES !

$H(e^{j\frac{5\pi}{4}}) = 1{,}8734 - j0{,}3333$ $\quad H(e^{j\frac{3\pi}{4}})$

$H(e^{j\frac{3\pi}{2}}) = 2{,}0769 - j1{,}3846$ $\quad H(e^{j\frac{\pi}{2}})$

$H(e^{j\frac{7\pi}{4}}) = -0{,}8859 - j0{,}6426$ $\quad H(e^{j\frac{\pi}{4}})$

FREQUENCY RESPONSE: $H(\omega) = [\,0 \quad -0{,}8859 + j0{,}6426 \quad 2{,}0769 + j1{,}3846 \quad 1{,}8734 + j0{,}3333 \quad 1{,}8 \quad 1{,}8734 - j0{,}3333 \quad 2{,}0769 - j1{,}3846 \quad -0{,}8859 - j0{,}6426\,]$

AMPLITUDE RESPONSE: $\|H(\omega)\| = [\,0 \quad 1{,}09 \quad 2{,}50 \quad 1{,}90 \quad 1{,}8 \quad 1{,}90 \quad 2{,}50 \quad 1{,}09\,]$

$\sqrt{(Re)^2 + (Im)^2}$ $\qquad \sqrt{(-0{,}8859)^2 + (0{,}6426)^2} = 1{,}09$ ETC ...

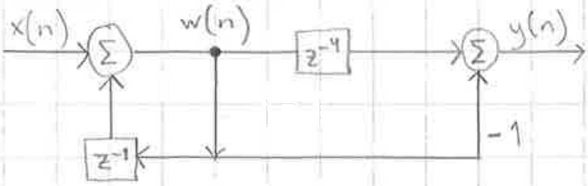
PHASE RESPONSE: $\phi(\omega) = [\,0° \quad 144{,}0° \quad 33{,}7° \quad 10{,}1° \quad 0° \quad -10{,}1° \quad -33{,}7° \quad -144{,}0°\,]$ DEG

$\arctan\left(\frac{Im}{Re}\right)\ (+k\cdot 180°)$

$k = \ldots, -1, 0, 1, \ldots$

$\phi(\omega) = [\,0 \quad 2{,}514 \quad 0{,}588 \quad 0{,}176 \quad 0 \quad -0{,}176 \quad -0{,}588 \quad -2{,}514\,]$ RAD

PROBLEM 2



$$\begin{cases} w(n) = x(n) + w(n-1) \\ y(n) = w(n-4) - w(n) \end{cases}$$

$$W(z) = X(z) + W(z)z^{-1}$$
$$Y(z) = W(z)z^{-4} - W(z) \Rightarrow Y(z) = W(z)(z^{-4} - 1)$$
$$W(z) = \frac{Y(z)}{z^{-4}-1}$$

$$\begin{cases} W(z) = X(z) + W(z)z^{-1} \\ W(z) = \dfrac{Y(z)}{z^{-4}-1} \end{cases}$$

$$\Rightarrow \frac{Y(z)}{z^{-4}-1} = X(z) + \frac{Y(z)z^{-1}}{z^{-4}-1}$$

$$\frac{Y(z) - Y(z)z^{-1}}{z^{-4}-1} = X(z)$$

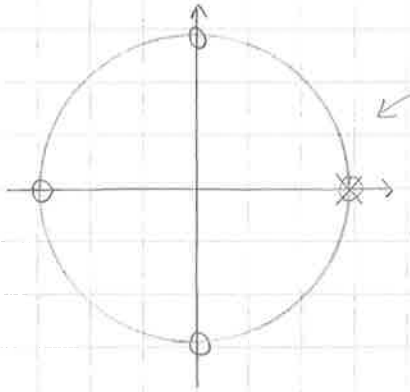
$$\frac{Y(z)(1-z^{-1})}{z^{-4}-1} = X(z) \Rightarrow \frac{X(z)}{Y(z)} = \frac{1-z^{-1}}{z^{-4}-1}$$

DIFFERENCE EQUATION:

$$X(z)(z^{-4}-1) = Y(z)(1-z^{-1})$$
$$X(z)z^{-4} - X(z) = Y(z) - Y(z)z^{-1}$$
$$x(n-4) - x(n) = y(n) - y(n-1)$$

$$\Rightarrow \underline{\underline{y(n) = x(n-4) - x(n) + y(n-1)}}$$

TRANSFER FUNCTION: $H(z) = \dfrac{Y(z)}{X(z)} = \underline{\underline{\dfrac{z^{-4}-1}{1-z^{-1}}}}$ THIS IS FINE!



$$= \frac{(z^{-2}-1)(z^{-2}+1)}{1-z^{-1}} = \frac{(z^{-1}-1)(z^{-1}+1)(z^{-2}+1)}{1-z^{-1}}$$

$$= \frac{-(1-z^{-1})(z^{-1}+1)(z^{-2}+1)}{1-z^{-1}}$$

$$= -(z^{-1}+1)(z^{-2}+1)$$

$$= -z^{-3} - z^{-2} - z^{-1} - 1 = -1 - z^{-1} - z^{-2} - z^{-3}$$