DIGITAL FILTERS - WEEK EXAM 2

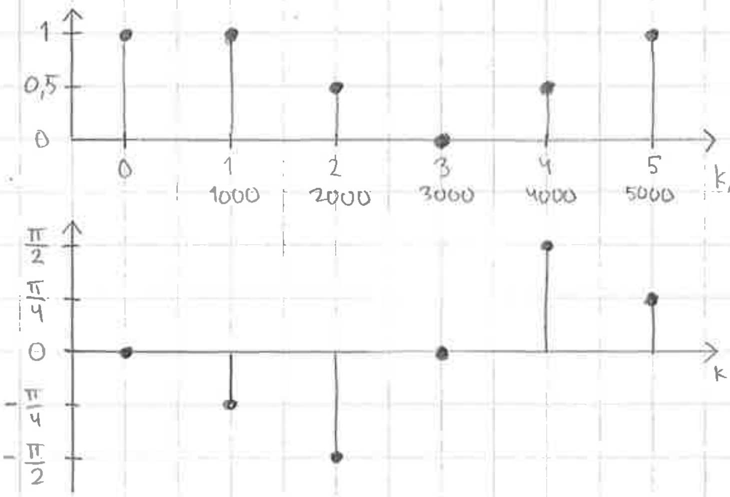
PROBLEM 1

$F_s = 6\ kHz$

SAMPLES : $H(f) = \{ 1\angle 0 \quad 1\angle -\frac{\pi}{4} \quad 0{,}5\angle -\frac{\pi}{2} \quad 0\angle 0 \quad 0{,}5\angle \frac{\pi}{2} \quad 1\angle \frac{\pi}{4} \}$

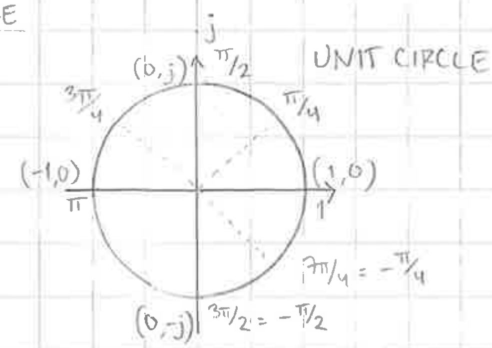
$\Rightarrow N = 6 \qquad = \{ e^{-j0} \quad e^{-j\frac{\pi}{4}} \quad 0{,}5\, e^{-j\frac{\pi}{2}} \quad 0 \quad 0{,}5\, e^{j\frac{\pi}{2}} \quad e^{j\frac{\pi}{4}} \}$

PLOTS :



SAMPLING FREQUENCY

$F_s = 6000\ Hz$



Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta \quad ; \quad e^{-j\theta} = \cos\theta - j\sin\theta$

$\cos 0 = 1$
$\sin 0 = 0$
$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\cos\frac{\pi}{2} = 0$
$\sin\frac{\pi}{2} = j$

$$\Rightarrow X(f) = \{ 1(\cos 0 + j\sin 0) \quad 1(\cos\tfrac{\pi}{4} - j\sin\tfrac{\pi}{4}) \quad 0{,}5(\cos\tfrac{\pi}{2} - j\sin\tfrac{\pi}{2}) \quad 0 \quad 0{,}5(\cos\tfrac{\pi}{2} + j\sin\tfrac{\pi}{2}) \quad 1(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}) \}$$

WE GET THE FOLLOWING SEQUENCE

$$X(k) = \{ 1 + j0 \quad \tfrac{\sqrt{2}}{2} - j\tfrac{\sqrt{2}}{2} \quad -j0{,}5 \quad 0 \quad j0{,}5 \quad \tfrac{\sqrt{2}}{2} + j\tfrac{\sqrt{2}}{2} \}$$

IDFT : $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{jnk\Omega}$, $\Omega = \frac{2\pi}{NT} = \frac{2\pi}{6\cdot 1} = \frac{2\pi}{6} = \frac{\pi}{3}$ (WE KNOW N=6 ASSUME T=1)

IDFT FOR THE SEQUENCE :

$$x(n) = \frac{1}{6}\sum_{k=0}^{5} X(k)\, e^{jnk\frac{\pi}{3}}$$

$$x(0) = \frac{1}{6}\sum_{k=0}^{5} X(k)\, e^{j\cdot \overset{n=0}{0}\cdot k\cdot\frac{\pi}{3}} = \frac{1}{6}\sum_{k=0}^{5} X(k)\, \overbrace{e^{j0}}^{=1}$$

$$= \frac{1}{6}\left(1 + \frac{\sqrt{2}}{2} \cancel{- j\frac{\sqrt{2}}{2}} \cancel{- j0{,}5} + 0 \cancel{+ j0{,}5} + \frac{\sqrt{2}}{2} \cancel{+ j\frac{\sqrt{2}}{2}}\right)$$

$$= \frac{1}{6}\left(1 + \frac{\sqrt{2}}{2}\cdot 2\right) = \frac{1}{6}(1 + \sqrt{2}) \approx 0{,}402 \quad \text{(FIRST COEFFICIENT)}$$

SECOND COEFFICIENT:

$$x(n) = \frac{1}{6}\sum_{k=0}^{5} \underline{X}(k)\, e^{j\cdot n\cdot k\cdot \frac{\pi}{3}}$$

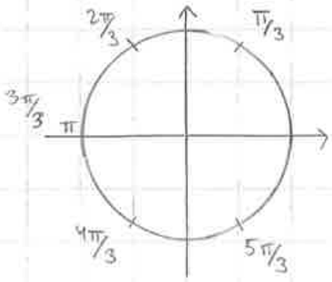
$$x(1) = \frac{1}{6}\sum_{k=0}^{5} \underline{X}(k)\, e^{j\cdot 1\cdot k\cdot \frac{\pi}{3}} = \frac{1}{6}\sum_{k=0}^{5} \underline{X}(k)\, e^{jk\frac{\pi}{3}}$$

$$= \frac{1}{6}\Big(\underbrace{\underline{X}(0)}_{=1}\underbrace{e^{j0\frac{\pi}{3}}}_{=1} + \underline{X}(1)\, e^{j\cdot 1\cdot\frac{\pi}{3}} + \underline{X}(2)\, e^{j\frac{2\pi}{3}} + \underline{X}(3)\, e^{j\pi} + \underline{X}(4)\, e^{j\frac{4\pi}{3}} + \underline{X}(5)\, e^{j\frac{5\pi}{3}}\Big)$$

$$= \frac{1}{6}\Big(1 + \Big(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\Big) e^{j\frac{\pi}{3}} - j0{,}5\, e^{j\frac{2\pi}{3}} + 0\cdot e^{j\pi} + j0{,}5\, e^{j\frac{4\pi}{3}} + \Big(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\Big) e^{j\frac{5\pi}{3}}\Big)$$

$$= \frac{1}{6}\Big(1 + \frac{\sqrt{2}}{2} e^{j\frac{\pi}{3}} - j\frac{\sqrt{2}}{2} e^{j\frac{\pi}{3}} - j0{,}5\, e^{j\frac{2\pi}{3}} + j0{,}5\, e^{j\frac{4\pi}{3}} + \frac{\sqrt{2}}{2} e^{j\frac{5\pi}{3}} + j\frac{\sqrt{2}}{2} e^{j\frac{5\pi}{3}}\Big).$$

$$e^{j\theta} = \cos\theta + j\cdot\sin\theta \quad ; \quad \text{e.g. } e^{j\frac{\pi}{3}} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = \overbrace{0{,}5}^{\cos\frac{\pi}{3}} + j\overbrace{\frac{\sqrt{3}}{2}}^{\sin\frac{\pi}{3}}$$



$$\cos\frac{\pi}{3} = \cos\frac{5\pi}{3} = 0{,}5$$
$$\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -0{,}5$$
$$\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{4\pi}{3} = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$$

cos: $\longrightarrow$

sin: $\uparrow$

NOW:

$$x(1) = \frac{1}{6}\Big(1 + \frac{\sqrt{2}}{2}\Big(0{,}5 + j\frac{\sqrt{3}}{2}\Big) - j\frac{\sqrt{2}}{2}\Big(0{,}5 + j\frac{\sqrt{3}}{2}\Big) - j0{,}5\Big(-0{,}5 + j\frac{\sqrt{3}}{2}\Big)$$
$$+ j0{,}5\Big(-0{,}5 - j\frac{\sqrt{3}}{2}\Big) + \frac{\sqrt{2}}{2}\Big(0{,}5 - j\frac{\sqrt{3}}{2}\Big) + j\frac{\sqrt{2}}{2}\Big(0{,}5 - j\frac{\sqrt{3}}{2}\Big)\Big)$$

$$= \frac{1}{6}\Big(1 + 0{,}5\cdot\frac{\sqrt{2}}{2} + j\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\cdot 0{,}5 - j\frac{\sqrt{2}}{2}\cdot j\frac{\sqrt{3}}{2} + j0{,}25 - j0{,}5\cdot j\frac{\sqrt{3}}{2}$$
$$- j0{,}25 + j0{,}5\cdot\Big(-j\frac{\sqrt{3}}{2}\Big) + \frac{\sqrt{2}}{2}\cdot 0{,}5 + \frac{\sqrt{2}}{2}\cdot\Big(-j\frac{\sqrt{3}}{2}\Big) + j\frac{\sqrt{2}}{2}\cdot 0{,}5 + j\frac{\sqrt{2}}{2}\cdot\Big(-j\frac{\sqrt{3}}{2}\Big)\Big) \quad \Big|\ j\cdot j = -1$$

REARRANGED AND $(j\cdot j = -1)$

$$= \frac{1}{6}\Big(1 + 0{,}5\cdot\frac{\sqrt{2}}{2} + 0{,}5\frac{\sqrt{2}}{2} + j\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - j\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\cdot 0{,}5 + j\frac{\sqrt{2}}{2}\cdot 0{,}5$$
$$+ \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + j0{,}25 - j0{,}25 + 0{,}5\cdot\frac{\sqrt{3}}{2} + 0{,}5\frac{\sqrt{3}}{2}\Big)$$

$$= \frac{1}{6}\Big(1 + 0{,}5\frac{\sqrt{2}}{2} + 0{,}5\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + 0{,}5\frac{\sqrt{3}}{2} + 0{,}5\frac{\sqrt{3}}{2}\Big)$$
$$= \frac{1}{6}\Big(1 + \frac{\sqrt{2}}{2} + \sqrt{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\Big)$$
$$\approx 0{,}633$$

$$\Rightarrow x(n) = \Big\{\frac{1}{6}\big(1 + \sqrt{2}\big) \quad \frac{1}{6}\Big(1 + \frac{\sqrt{2}}{2} + \sqrt{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\Big) \quad \ldots\Big\}$$
$$\approx \{0{,}402 \quad 0{,}633 \quad \ldots\}$$

COMPLEX VALUES CANCEL OUT !

PROBLEM 2

ZEROS: $z_0 = 0{,}95 + 0{,}35j$, $z_1 = 0{,}95 - 0{,}35j$

POLES: $z_{p0} = 0{,}9 + 0{,}3j$, $z_{p1} = 0{,}9 - 0{,}3j$

IMPULSE RESPONSE ? $\rightarrow$ TRANSFER FUNCTION $H(z)$

$$H(z) = \frac{(z - z_0)(z - z_1)}{(z - z_{p0})(z - z_{p1})} = \frac{(z - 0{,}95 - 0{,}35j)(z - 0{,}95 + 0{,}35j)}{(z - 0{,}9 - 0{,}3j)(z - 0{,}9 + 0{,}3j)}$$

$$= \frac{z^2 - 0{,}95z + 0{,}35jz - 0{,}95z + 0{,}9025 - 0{,}3325j - 0{,}35jz + 0{,}3325j + 0{,}1225}{z^2 - 0{,}9z + 0{,}3jz - 0{,}9z + 0{,}81 - 0{,}27j - 0{,}3jz + 0{,}27j + 0{,}09}$$

$$= \frac{z^2 - 1{,}9z + 1{,}025}{z^2 - 1{,}8z + 0{,}9}$$

THREE COEFFICIENTS SUFFICE

LONG DIVISION!

$$\begin{array}{r|l} & 1 - 0{,}1z^{-1} - 0{,}055z^{-2} \\ \hline z^2 - 1{,}8z + 0{,}9 & z^2 - 1{,}9z + 1{,}025 \\ & -(z^2 - 1{,}8z + 0{,}9) \\ \hline & -0{,}1z + 0{,}125 \\ & -(-0{,}1z + 0{,}18 - 0{,}09z^{-1}) \\ \hline & -0{,}055 + 0{,}09z^{-1} \quad \text{ETC.} \end{array}$$

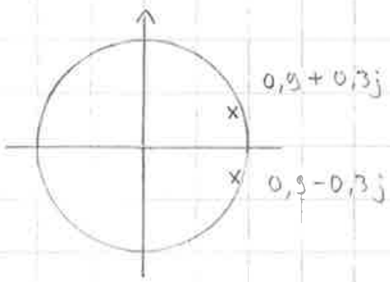
$$H(z) = 1 - 0{,}1z^{-1} - 0{,}055z^{-2} \ldots$$

$\Rightarrow$ IMPULSE RESPONSE

$$h(n) = \{1 \quad -0{,}1 \quad -0{,}055 \quad \ldots\}$$

IS THE FILTER STABLE? (= ARE THE POLES INSIDE THE UNIT CIRCLE?)

POLES: $z_{p0} = 0{,}9 + 0{,}3j$, $z_{p1} = 0{,}9 - 0{,}3j$



DISTANCE FROM ORIGIN

$$\sqrt{0{,}9^2 + 0{,}3^2} = \sqrt{0{,}81 + 0{,}09}$$

$$= \sqrt{0{,}9} = 0{,}948683 < 1$$

$\Rightarrow$ THE FILTER IS STABLE!